THE WEDDERBURN STRUCTURE THEOREMS

ROY DUBISCH, Syracuse University

1. Introduction. Considerable effort is being currently expended in the development of a "structure" theory for non-associative algebras [see, for example, references **2, 3**] so that a description of the structure theory for associative algebras may be helpful at this time in picturing possible trends of research in the non-associative case.

Briefly, the Wedderburn structure theory for associative algebras (henceforth called algebras) first divides all algebras into two classes, namely, *nilpotent* algebras and *non-nilpotent* algebras (these terms and others used in this section will be defined later). The classification of nilpotent algebras is still one of the outstanding problems of algebra [see **1**, p. 172] although a number of special results have been obtained. For the non-nilpotent algebras, however, we may continue and consider the *radical* of such an algebra which is, in itself, a nilpotent algebra. Then, by considering the algebra *modulo* its radical, we arrive at an algebra called a *semi-simple* algebra whose radical is the *zero ideal*. It is then possible to express each semi-simple algebra as the *direct sum* of *simple* algebras and these, finally, can be expressed as the *direct product* of a *total matric* algebra and a *division* algebra.

Thus we see that the study of algebras is equivalent to the study of (1) nilpotent algebras, (2) total matric algebras, and (3) division algebras. It should be realized, of course, that this is a gross over-simplification of the study of algebras but it is at least a fundamental basis for further work. It should also be remarked that some of the definitions given below are somewhat loose but should, nevertheless, give a clear idea of the concepts involved.

2. Definition of an algebra. An algebra is always considered *over* a *field*, which, for simplicity of statement of the structure theorems, we will consider to be a non-modular field. This, the reader may, without any great loss in generality, consider to be the field of rational numbers. The theory of algebras over a *modular* field, such as the field of all integers modulo a prime p, is quite similar but requires a knowledge of certain additional concepts which are well to be avoided in a first survey of this topic.

An algebra $\mathfrak{A}$, then, of order n over $\mathfrak{F}$, consists of all linear combinations,

$$a = \alpha_1 u_1 + \cdots + \alpha_n u_n = \sum \alpha_i u_i,$$

of n basal elements $u_1, \cdots, u_n$ with coefficients $\alpha_i \epsilon \mathfrak{F}$. The sum of two such elements is defined by

$$(\sum \alpha_i u_i) + (\sum \beta_i u_i) = \sum (\alpha_i + \beta_i) u_i, \qquad (i = 1, \cdots, n),$$

while the scalar product of an element a of $\mathfrak{A}$ by an element β of $\mathfrak{F}$ is given by

$$\beta(\sum \alpha_i u_i) = \sum (\beta \alpha_i) u_i \qquad (i = 1, \cdots, n).$$

Finally, the product of any two elements a and b of $\mathfrak{A}$ is determined by

$$(\sum \alpha_i u_i)(\sum \beta_j u_j) = \sum (\alpha_i \beta_j) u_i u_j \qquad (i, j = 1, \cdots, n),$$

and a table giving the product $u_i u_j$ of any two basal elements of the algebra as some element (not necessarily a basal element) of the algebra. Naturally it is also demanded that multiplication be associative.

Thus, an algebra is determined by the multiplication table of its basal elements and, as certain of the examples we now give show, we are not assuming the existence of an identity e for an arbitrary algebra ($ea = ae = a$ for all $a \epsilon \mathfrak{A}$); we are not assumng commutivity ($ab = ba$); and we do not assert that $ab = 0$ implies that a or b is zero.

EXAMPLE 1: The algebra of complex numbers with a basis 1, i over the real numbers with $i^2 = -1$. This is also a field. Note that there is no unique set of basal elements as 1, $1-i$ is also a set of basal elements.

EXAMPLE 2: Let $\mathfrak{A}$ have a basis u_1, u_2 over any field $\mathfrak{F}$ with $u_1^2 = u_2^2 = u_1 u_2 = u_2 u_1 = 0$. This is an example of a *zero* algebra (that is, an algebra for which $ab = 0$ for every a and b of the algebra). The general quantity of $\mathfrak{A}$ is given by $a = \alpha_1 u_1 + \alpha_2 u_2$ and $ab = (\alpha_1 u_1 + \alpha_2 u_2)(\beta_1 u_1 + \beta_2 u_2) = \alpha_1 \beta_1 u_1^2 + \alpha_1 \beta_2 u_1 u_2 + \alpha_2 \beta_1 u_2 u_1 + \alpha_2 \beta_2 u_2^2 = 0$.

EXAMPLE 3: Let $\mathfrak{A}$ have the basis 1, i, j, k, over the real numbers with the *multiplication table*

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

This is the algebra of quaternions. Note that 1 is a unity, that is, $1 \cdot a = a \cdot 1 = a$ for $a = \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 k$. Furthermore $ij = -ji$ so that it is not a *commutative* algebra and, for every non-zero $a \epsilon \mathfrak{A}$, there exists a $b \epsilon \mathfrak{A}$ such that $ab = ba = 1$ or, equivalently, as may be shown, $ab = 0$ implies a or b equal to zero. That is, $\mathfrak{A}$ is a division algebra. But if we consider $\mathfrak{A}$ over the field of complex numbers $\mathfrak{C}$ where $\zeta^2 = -1$ we have $(\zeta - i)(\zeta + i) = \zeta^2 - i^2 = -1 - (-1) = 0$ so that $\mathfrak{A}$ is not a division algebra over $\mathfrak{C}$. We say that $\mathfrak{C}$ *splits* $\mathfrak{A}$.

EXAMPLE 4: The set of all 2×2 (or $n \times n$) matrices over any field. This is called a *total matric* algebra. A basis (*ordinary matric* basis) is given by

$$u_1 = \begin{pmatrix} 10 \\ 00 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 01 \\ 00 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} 00 \\ 10 \end{pmatrix}, \qquad u_4 = \begin{pmatrix} 00 \\ 01 \end{pmatrix}.$$

Note that $u_2^2 = 0$ and that $u_1 + u_4$ is the *identity* matrix.

3. Ideals. In any algebra $\mathfrak{A}$ we may have subsets $\mathfrak{B}$ of $\mathfrak{A}$ which have the property that, for every $b \epsilon \mathfrak{B}$ and any $a \epsilon \mathfrak{A}$, the products ab and ba are in $\mathfrak{B}$ and, for every b_1 and $b_2 \epsilon \mathfrak{B}$, the sum $b_1 + b_2$ is in $\mathfrak{B}$. Such a subset is called an *ideal* of $\mathfrak{A}$.

If the requirement that ba and ab be in $\mathfrak{B}$ is weakened by specifying it only for $a\epsilon\mathfrak{B}$, we call $\mathfrak{B}$ a sub-algebra of $\mathfrak{A}$. Thus every ideal of $\mathfrak{A}$ is also a sub-algebra of $\mathfrak{A}$, but as shown in Example 6 below, the converse is not necessarily true.

EXAMPLE 5: In Example 2, the set $\mathfrak{B}$ of all αu_1 is an ideal. For $(\alpha u_1)(\beta u_1+\gamma u_2)=0=0\cdot u_1$ is in $\mathfrak{B}$ and $\alpha u_1+\beta u_1=(\alpha+\beta)u_1$ is in $\mathfrak{B}$.

EXAMPLE 6: The only ideals of the algebra of quaternions are the set of all multiples of 1 (that is, the entire algebra) and the *zero* ideal consisting of all multiples of zero. For, if any ideal $\mathfrak{B}$ contains 1, it clearly contains $1\cdot i$, $1\cdot j$, $1\cdot ij=1\cdot k$, and hence is equal to $\mathfrak{A}$, since it contains a basis of $\mathfrak{A}$. If it contains i it contains $i\cdot i=-1$ and hence is again equal to $\mathfrak{A}$, and so on. But there are several sub-algebras of $\mathfrak{A}$ as, for example, the set of all $\alpha\cdot 1+\beta\cdot i$ which is *equivalent* to the algebra of complex numbers.

It may similarily be shown that the only ideals of the algebra of all $n\times n$ matrices are the algebra itself and the zero ideal. Such algebras, if they are not zero algebras of order 1, are called *simple* algebras. That is, a simple algebra is an algebra which is not a zero algebra of order 1 and which has no ideals except the algebra itself and the zero ideal; that is, it has no *proper* ideals.

4. The radical. An element $x\neq 0$ of an algebra is said to be *nilpotent* if some power of x is zero. It is said to be *properly nilpotent* if every multiple ax or xa of x is nilpotent or zero. The set of all properly nilpotent elements of an algebra $\mathfrak{A}$ may be shown to be an ideal [3, pp. 22–23, or 5], and is called the *radical* of $\mathfrak{A}$.

EXAMPLE 7: The radical of the algebra $\mathfrak{A}$ of Example 2 is $\mathfrak{A}$ itself.

EXAMPLE 8: The quantities u_2 and u_3 of the algebra of Example 4 are clearly nilpotent, but there are no properly nilpotent elements except zero.

EXAMPLE 9: Consider the algebra $\mathfrak{A}$ with a basis e_1, e_2, u and with multiplication table given by

	e_1	e_2	u
e_1	e_1	0	u
e_2	0	e_2	0
u	θ	u	0

Since $e_1^2=e_1$, e_1 is certainly not nilpotent. Similarly e_2 is not nilpotent. But u is a properly nilpotent element since $u\ (\alpha_1e_1+\alpha_2e_2+\alpha_3u)=\alpha_1u$ and $(\alpha_1u)^2=0$. Hence the radical $\mathfrak{N}$ of $\mathfrak{A}$ contains all αu. If $a=\alpha e_1+\beta e_2+\gamma u$ is also in $\mathfrak{N}$, $e_1a-\gamma u=\alpha e_1$ is in $\mathfrak{N}$ (see definition of an ideal) which is a contradiction, since e_1 is not nilpotent, unless $\alpha=0$. But then $ae_2-\gamma u=\beta e_2$ is in $\mathfrak{N}$, which is a contradiction unless $\beta=0$. Hence $\mathfrak{N}$ consists of all αu.

5. Difference algebras. Consider an algebra $\mathfrak{A}$ with an ideal $\mathfrak{B}$. *The difference algebra* $\mathfrak{A}-\mathfrak{B}$ (analogous to a quotient group in group theory) consists of elements $[a]$ defined for every $a\epsilon\mathfrak{A}$. Explicitly, $[a]$ is the set of all elements $a+b$ for any $b\epsilon\mathfrak{B}$ where we define $[a]=[c]$ to mean that $a-c$ is in $\mathfrak{B}$. Furthermore, we define $[a]+[d]=[a+d]$, $[a][d]=[ad]$ and may then show that $\mathfrak{A}-\mathfrak{B}$ is an

algebra by virtue of the fact that $\mathfrak{A}$ is an algebra and $\mathfrak{B}$ is an ideal of $\mathfrak{A}$.

EXAMPLE 10: In the algebra $\mathfrak{A}-\mathfrak{N}$, for $\mathfrak{A}$ and $\mathfrak{N}$ defined in Example 9, we have a basis $[e_1]$, $[e_2]$ with $[e_1][e_1]=[e_1e_1]=[e_1]$; $[e_1][e_2]=[e_1e_2]=[0]$, $[e_2][e_1]=[e_2e_1]=[0]$, $[e_2e_2]=[e_2^2]=[e_2]$.

6. Sums and direct sums. Consider an algebra $\mathfrak{A}$ with sub-algebras $\mathfrak{B}$ and $\mathfrak{C}$. We say that $\mathfrak{A}$ is the *supplementary sum* of $\mathfrak{B}$ and $\mathfrak{C}$, and write $\mathfrak{A}=\mathfrak{B}+\mathfrak{C}$, if every $a\epsilon\mathfrak{A}$ is uniquely expressible as $a=b+c$ for $b\epsilon\mathfrak{B}$ and $c\epsilon\mathfrak{C}$.

EXAMPLE 11: The total matric algebra $\mathfrak{A}$ of Example 4 has the sub-algebra $\mathfrak{B}$ with a basis u_1, u_2 and a sub-algebra $\mathfrak{C}$ with a basis u_3, u_4 and $\mathfrak{A}=\mathfrak{B}+\mathfrak{C}$.

Now consider two algebras $\mathfrak{A}$ and $\mathfrak{B}$. The *direct sum*, $\mathfrak{A}\oplus\mathfrak{B}$, of $\mathfrak{A}$ and $\mathfrak{B}$ is defined as follows: Let $\mathfrak{A}$ have a basis $u_1, \cdots, u_n$ and $\mathfrak{B}$ a basis $v_1, \cdots, v_m$. Then $\mathfrak{A}\oplus\mathfrak{B}$ is the algebra with a basis $u_1, \cdots, u_n, v_1, \cdots, v_m$ and a multiplication table defined by the multiplication tables of $\mathfrak{A}$ and $\mathfrak{B}$ and $u_iv_j=v_ju_i=0$, $(i=1, \cdots, n; j=1, \cdots, m)$. Then $\mathfrak{A}\oplus\mathfrak{B}\oplus\mathfrak{C}$ is defined as the direct sum of $\mathfrak{A}\oplus\mathfrak{B}$ and $\mathfrak{C}$, and so on.

EXAMPLE 12: The direct sum of the algebras of Examples 1 and 2 has a basis $(1, i, u_1, u_2)$, and multiplication table given by

	1	i	u_1	u_2
1	1	i	0	0
i	i	-1	0	0
u_1	0	0	0	0
u_2	0	0	0	0

7. Direct products. The final concept necessary before stating the Wedderburn structure theorems is that of a *direct product* of two algebras $\mathfrak{A}$ and $\mathfrak{B}$. Let $u_1, \cdots, u_n$ be a basis of $\mathfrak{A}$ and $v_1, \cdots, v_m$ be a basis of $\mathfrak{B}$. Then the direct product $\mathfrak{A}\times\mathfrak{B}$ is an algebra of order mn with a basis $u_1\cdot v_1, u_1\cdot v_2, \cdots, u_1\cdot v_m, \cdots, u_n\cdot v_1, \cdots, u_n\cdot v_m$ with the multiplication defined by the multiplication of $\mathfrak{A}$ and $\mathfrak{B}$ and the agreement that the symbolic product $u_i\cdot v_j$ shall be the same as the product $v_j\cdot u_i$.

EXAMPLE 12: The direct product of the algebras of Examples 1 and 2 over the field of real numbers has a basis given by $(1\cdot u_1, 1\cdot u_2, i\cdot u_1, i\cdot u_2)$, and is again a zero algebra.

EXAMPLE 13: The most important type of direct product for our purposes is that of a total matric algebra and a division algebra. This may be considered to be a total matric algebra with *elements which are those of the division algebra.* Thus the direct product of the algebras of Example 1 and Example 4 may be taken to have a basis,

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}i&0\\0&0\end{pmatrix}, \begin{pmatrix}0&i\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\i&0\end{pmatrix}, \begin{pmatrix}0&0\\0&i\end{pmatrix}.$$

8. The principal* Wedderburn theorem.

Let $\mathfrak{A}$ be an algebra with a radical $\mathfrak{N}$. Then $\mathfrak{A}-\mathfrak{N}$ has a zero radical and $\mathfrak{A}=\mathfrak{S}+\mathfrak{N}$, where $\mathfrak{S}$ is equivalent to $\mathfrak{A}-\mathfrak{N}$.

This, then, reduces our consideration of an arbitrary algebra to the consideration of nilpotent algebras ($\mathfrak{N}$) and algebras with zero radicals ($\mathfrak{S}$).

EXAMPLE 15: In Example 10 it is easy to see that $\mathfrak{A}-\mathfrak{N}$ has a zero radical and that $\mathfrak{A}=\mathfrak{S}+\mathfrak{N}$, where $\mathfrak{S}$ is the algebra with a basis $[e_1]$, $[e_2]$.

9. The first Wedderburn structure theorem. We define a *semi-simple* algebra as one whose radical is the zero ideal, and we have the following theorem:

Every semi-simple algebra is uniquely expressible (except for order) as a direct sum of simple sub-algebras. Conversely, every direct sum of simple algebras is semi-simple.

Hence we reduce our study of arbitrary algebras to a study of simple algebras and nilpotent algebras.

EXAMPLE 16: In Example 10 we have $\mathfrak{S}=\mathfrak{F}_1\oplus\mathfrak{F}_2$, where $\mathfrak{F}_1$ is the algebra with basis $[e_1]$ and $\mathfrak{F}_2$ is the algebra with basis $[e_2]$; actually, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are both fields.

10. The second Wedderburn structure theorem.

Every simple algebra $\mathfrak{A}$ is expressible as $\mathfrak{A}=\mathfrak{M}\times\mathfrak{D}$, where $\mathfrak{M}$ is a total matric algebra and $\mathfrak{D}$ is a division algebra.

Hence, as described in our introduction, we have reduced the study of arbitrary associative algebras to the study of nilpotent algebras, division algebras, and total matric algebras. Of these types, total matric algebras may be considered to be completely described as is evident from their definition, the structure of division algebras is completely determined over special fields at least (notably the field of rational numbers) while the classification of nilpotent algebras is still to a great extent an open question.

11. Non-associative theory. Just a few years ago it would have been possible to conclude any discussion of associative algebras with a *short* summary of most of the work done in non-associative algebras (algebras satisfying the requirements given for associative algebras except for the associative law) except for certain *special* types of non-associative algebras, notably the *Lie* algebras. However, this is no longer the case. So we shall content ourselves here with a brief mention of some of the difficulties encountered in attempting to parallel the non-associative theory with the associative.

Basically it would seem that a considerable part of the difficulty is concerned with a proper re-phrasing of the definitions of the various concepts occurring in the associative theory. For example, the concept of nilpotency needs to be redefined, for a^n is no longer necessarily meaningful since it may be that $a(aa)\neq(aa)a$. Hence the concept of a radical as originally defined is no longer

* The words "principal," "first," and "second," as used in the headings of Sections 8, 9, and 10, do not imply any classification of the Wedderburn theorems with respect to importance, but are merely the customary designations for these theorems.

applicable. Several alternative definitions have been proposed of which we mention two. The first, due to Albert [4], very ingeniously avoids the difficulty, so to speak, by defining a semi-simple algebra as an algebra which is a direct sum of simple algebras (and the latter concept may be defined without change in the non-associative theory), and then defines the radical as the ideal $\mathfrak{N}$ of $\mathfrak{A}$ for which $\mathfrak{A}-\mathfrak{N}$ is semi-simple and $\mathfrak{N}$ is contained in every ideal $\mathfrak{B}$ of $\mathfrak{A}$ for which $\mathfrak{A}-\mathfrak{B}$ is semi-simple ($\mathfrak{N}$ is the "least" such ideal).

On the other hand, Jacobson [7] (see also Perlis [11]) first calls an element z of an algebra $\mathfrak{A}$, *right quasi-regular* if there exists an element $z' \epsilon \mathfrak{A}$ such that $z'+z+zz'=0$. He then defines a *right* ideal $\mathfrak{B}$ of $\mathfrak{A}$ (ba but not necessarily $ab \epsilon \mathfrak{B}$ for every $b \epsilon \mathfrak{B}$ and $a \epsilon \mathfrak{A}$) to be quasi-regular if all the elements of $\mathfrak{B}$ are right quasi-regular. The radical is then the sum of all the quasi-regular right ideals of $\mathfrak{A}$.

When these definitions of the radical are applied to the associative case they coincide with the definitions we have given for the radical in terms of properly nilpotent elements. This is also true for alternative algebras [6], but some curious results are obtained in the general non-associative case. Thus Albert [4] exhibits a non-associative algebra whose radical (in his sense) is a field—about as far from a set of properly nilpotent elements as one could image!

Another definition needing re-tooling for non-associative algebras is that of a total matric algebra [see, for example, 3], but the concept of a division algebra ($ab \neq 0$ unless a or b is zero) seems to be as satisfactory in non-associative theory as it is in associative theory.

Finally, we may mention that for certain types of non-associative algebras, the difficulties expressed here are more easily overcome, and more or less complete structure theories have been worked out for the *alternative* algebras [12, 13], *Lie* algebras [8, 9] and *Jordan* algebras [3, 10]. These algebras replace the associative law by some weakened form of it. Thus, in the alternative algebras, we have $a(xx)=(ax)x$ and $(xx)a=x(xa)$ (Example: the *Cayley* numbers); in the Lie algebras, $a(bc)+b(ca)+c(ab)=0$ (Example: the set of all $n \times n$ matrices under the multiplication defined by $A \times B = AB - BA$ where AB and BA indicate the ordinary matric products); and in the Jordan algebras we have $x(yx^2)=(xy)x^2$ (Example: the set of all $n \times n$ matrices under the multiplication defined by $A \times B = AB + BA$).

Bibliography

(This is only a partial list of references to topics discussed. Further bibliographies are given in 1, 4, 7, 8 and 9.)

1. A. A. Albert, Structure of Algebras, American Mathematical Society Colloquium Publications, vol. XXIV, New York 1939.

2. A. A. Albert, Non-associative Algebras. I. Fundamental Concepts and Isotopy, Annals of Mathematics, vol. 43, 1942, pp. 685–707.

3. A. A. Albert, On Jordan Algebras of Linear Transformation, Transactions of the American Mathematical Society, vol. 59, 1946, pp. 524–555.

4. A. A. Albert, The Radical of a Non-associative Algebra, Bulletin of the American Mathematical Society, vol. 48, 1942, pp. 891–897.

5. R. Brauer, On the Nilpotency of the Radical of a Ring, Bulletin of the American Mathematical Society, vol. 48, 1942, pp. 752–758.

6. Roy Dubisch and Sam Perlis, On the Radical of a Non-associative Algebra (abstract), Bulletin of the American Mathematical Society, vol. 50, 1944, p. 53.

7. N. Jacobson, The Radical and Semi-simplicity for Arbitrary Rings, American Journal of Mathematics, vol. 67, 1945, pp. 300–320.

8. N. Jacobson, Rational Methods in the Theory of Lie Algebras, Annals of Mathematics, vol. 36, 1935, pp. 987–991.

9. N. Jacobson, Simple Lie Algebras of Type A, Annals of Mathematics, vol. 39, 1938, pp. 181–188.

10. P. Jordan, J. von Neumann, and E. Wigner, On an Algebraic Generalization of the Quantum Mechanical Formalism, Annals of Mathematics, vol. 35, 1934, pp. 29–64.

11. Sam Perlis, A Characterization of the Radical of an Algebra, Bulletin of the American Mathematical Society, vol. 48, 1942, pp. 128–132.

12. J. H. M. Wedderburn, On a Theorem in Hypercomplex Numbers, Royal Society of Edinburgh. Proceedings, vol. 26 (1905), pp. 1–3.

13. J. H. M. Wedderburn, On Hypercomplex Numbers, London Mathematical Society. Proceedings, vol. 6 (1907), pp. 77–118.

14. M. Zorn, Theorie der Alternativer Ringe, Hamburg Mathematisches Seminar. Abhandlungen, vol. 8, 1930, pp. 123–147.

15. M. Zorn, Alternative Rings and Related Questions. I: Existence of the Radical, Annals of Mathematics, vol. 42, 1941, pp. 676–686.

UNION TORSION OF A CURVE ON A SURFACE

C. E. SPRINGER, University of Oklahoma

1. Introduction. The definitions of the well known geodesic curvature and geodesic torsion of a curve on a surface involve the notion of the congruence of normals to the surface. One may ask for the analogues of these definitions relative to a congruence of straight lines which are not normal to the surface.

In a previous paper [1] the author studied union curves and gave a definition of *union curvature* of a curve on a surface relative to a given rectilinear congruence. The purpose of the present note is to exhibit a definition of the *union torsion* of a curve on a surface. The union torsion reduces to the geodesic torsion for the case in which the congruence is normal to the surface. It will be shown that a union curve is a plane curve if, and only if, it is tangent to one of the curves in which the developables of the congruence intersect the surface.

The notation of Eisenhart [2] will be employed. Greek indices will always take the range 1, 2, and Latin indices the range 1, 2, 3. The summation convention of the tensor analysis as to repeated indices will be observed.

2. Analytical development. Let the surface be represented by $x^i = x^i(u^1, u^2)$, $(i = 1, 2, 3)$, referred to a rectangular cartesian system of coördinates. The functions $x^i(u^1, u^2)$ together with their partial derivatives to the second order, are to

be regarded as continuous at any point P of the surface under consideration. A unique line of the congruence is specified at each point P of the surface by the direction cosines

$$\lambda^i = \lambda^i(u^1, u^2), \qquad (\lambda^i\lambda^i = 1), \tag{1}$$

where the functions $\lambda^i(u^1, u^2)$, with their first partial derivatives, are continuous at the point P. The functions $\lambda^i(u^1, u^2)$ may be expressed in the form

$$\lambda^i = p^\alpha x^i{}_{,\alpha} + qX^i, \tag{2}$$

where p^α, $(\alpha = 1, 2)$, are the contravariant components of a surface vector at P, q is a positive scalar function, $x^i{}_{,\alpha}$ denotes the covariant derivative of x^i with respect to u^α based on the fundamental tensor $g_{\alpha\beta} = x^i{}_{,\alpha}x^i{}_{,\beta}$, and X^i are the direction cosines of the normal to the surface at P.

If θ is the angle between the lines with directions X^i and λ^i, multiplication of equations (2) by X^i shows that $q = \cos\theta$. For the angle ϕ between the tangent to a curve C: $u^\alpha = u^\alpha(s)$ through P on the surface and the line of the congruence through P with direction λ^i, we have

$$\cos\phi = \lambda^i \frac{dx^i}{ds} = (p^\alpha x^i{}_{,\alpha} + qX^i)x^i{}_{,\beta}u'^\beta = g_{\alpha\beta}p^\alpha u'^\beta, \tag{3}$$

where the prime indicates differentiation with respect to arc length s on C.

Covariant differentiation of λ^i, and use of the Gauss and Weingarten equations,

$$x^i{}_{,\alpha\beta} = d_{\alpha\beta}X^i, \qquad X^i{}_{,\beta} = -d_{\beta\gamma}g^{\gamma\sigma}x^i{}_{,\sigma},$$

lead to the expression

$$\lambda^i{}_{,\alpha} = \mu^\gamma{}_\alpha x^i{}_{,\gamma} + \nu_\alpha X^i, \tag{4}$$

where $\mu^\gamma{}_\alpha$ and ν_α are defined by

$$\mu^\gamma{}_\alpha \equiv p^\gamma{}_{,\alpha} - qd_{\alpha\sigma}g^{\sigma\gamma}, \qquad \nu_\alpha \equiv q_{,\alpha} + p^\beta d_{\alpha\beta}. \tag{5}$$

The curve C is a union curve relative to the congruence of lines with directions λ^i in case the osculating plane to C at every point P of C contains the line of the congruence at P. The differential equation of the union curves on the surface may be written in the form [1, p. 688]

$$e_{\sigma\tau}(p^\sigma K_n - q\rho^\sigma)u'^\tau = 0, \tag{6}$$

where $e_{12} = 1$, $e_{21} = -1$, $e_{11} = e_{22} = 0$, K_n is the normal curvature of C at P, and ρ^σ are the contravariant components of the curvature vector of C at P.

3. Torsion of a union curve. Let the direction cosines of the tangent, principal normal, and binormal to the curve C at P be denoted by α^i, β^i, γ^i. Then, the determinant $\epsilon_{ikl}\alpha^i\beta^k\gamma^l$ is equal to unity, and

$$\gamma^i = \epsilon_{ikl}\alpha^k\beta^l. \tag{7}$$

If C is a union curve, the principal normal to C lies in the plane of the lines with directions λ^i and $\alpha^i = dx^i/ds$. Hence

$$\beta^i = a\frac{dx^i}{ds} + b\lambda^i, \tag{8}$$

where a and b are to be determined. Multiplication of equations (8) in turn by dx^i/ds and β^i, and use of equation (3) yield $a = -\cot\phi$, $b = \csc\phi$, so that

$$\beta^i = \csc\phi\left(\lambda^i - \cos\phi\frac{dx^i}{ds}\right). \tag{9}$$

Use of β^i from (9) in (7) gives

$$\gamma^i = \csc\phi\epsilon_{ikl}\frac{dx^k}{ds}\left(\lambda^l - \cos\phi\frac{dx^l}{ds}\right) = \csc\phi\epsilon_{ikl}\frac{dx^k}{ds}\lambda^l. \tag{10}$$

By the Frenet formulas,

$$\frac{d\gamma^i}{ds} = \tau\beta^i, \tag{11}$$

where τ is the torsion of the curve C at P. Multiplication of equations (11) by β^i yields

$$\tau = \beta^i\frac{d\gamma^i}{ds}. \tag{12}$$

Use of β^i from (9) and of $d\gamma^i/ds$ as computed from (10) in (12) gives

$$\tau = \csc^2\phi\epsilon_{ikl}\left(\lambda^i - \cos\phi\frac{dx^i}{ds}\right)\left(-\cot\phi\frac{dx^k}{ds}\lambda^l\frac{d\phi}{ds} + \frac{dx^k}{ds}\frac{d\lambda^l}{ds} + \frac{d^2x^k}{ds^2}\lambda^l\right). \tag{13}$$

On dropping the vanishing determinants in (13), one finds

$$\tau = \csc^2\phi\epsilon_{ikl}\left(\lambda^i\frac{dx^k}{ds}\frac{d\lambda^l}{ds} - \cos\phi\frac{dx^i}{ds}\frac{d^2x^k}{ds^2}\lambda^l\right) \tag{14}$$

or, by a permutation of the indices i, k, l,

$$\tau = \csc^2\phi\epsilon_{ikl}\lambda^i\frac{dx^k}{ds}\left(\frac{d\lambda^l}{ds} - \cos\phi\frac{d^2x^l}{ds^2}\right). \tag{15}$$

Now, by use of equations (3), (4), and the fact that

$$\frac{d^2x^k}{ds^2} = \rho^\gamma x^k{}_{,\gamma} + K_n X^k,$$

the expression in parentheses in equation (15) can be written in the form

$$[(\mu^\gamma{}_\alpha - p_\alpha\rho^\gamma)x^l{}_{,\gamma} + (\nu_\alpha - p_\alpha K_n)X^l]u'^\alpha. \tag{16}$$

On using the expressions (2) and (16) in (15), and on observing that [2, pp. 135, 213]

$$\epsilon_{ikl}X^i x^k{}_{,\alpha}x^l{}_{,\beta} = e_{\alpha\beta}\sqrt{g} \equiv \epsilon_{\alpha\beta},$$

one finds that the formula for τ becomes

$$\tau = \csc^2\phi\epsilon_{\tau\alpha}[(p^\tau\nu_\beta - q\mu^\tau{}_\beta)u'^\alpha u'^\beta + p_\beta(qp^\tau - p^\tau K_n)u'^\alpha u'^\beta. \tag{17}$$

Because C is a union curve, equation (6) holds, and the formula for τ reduces to

$$\tau = \csc^2\phi\epsilon_{\tau\alpha}(p^\beta\nu_\beta - q\mu^{\tau\beta})u'^\alpha u'^\beta. \tag{18}$$

4. Union torsion of a curve. If the congruence is required to be normal to the surface, $p^\tau = 0$, $(\tau = 1, 2)$; $q = 1$, $\phi = \pi/2$, the union curve C becomes a geodesic curve on the surface, and τ becomes τ_g, the geodesic torsion of the geodesic curve through P. By equations (5), equation (18) becomes, in the present case of a normal congruence,

$$\tau_g = \epsilon_{\beta\alpha}d_{\beta\sigma}g^{\sigma\tau}u'^\alpha u'^\beta, \tag{19}$$

which is equivalent to the formula for τ_g as given by Eisenhart [2, p. 247].

Because the expression for τ in (18) depends only upon a point and a direction through the point on the surface, it is the same for all curves through a point with a common surface tangent. Thus, the *union torsion* of a curve C on a surface may be defined as the torsion of the union curve on the surface in the direction of the curve C.

The net of curves in which the developables of the congruence (2) intersect the surface is represented by the differential equation [3, p. 992]

$$\epsilon_{\alpha\tau}(p^\tau\nu_\beta - q\mu^\tau{}_\beta)du^\alpha du^\beta = 0. \tag{20}$$

The net given by equation (20) may be styled the *intersector net.*

On comparing equations (18) and (20), one concludes that *a union curve relative to a congruence is a plane curve if, and only if, it is tangent to a curve of the intersector net of the congruence.* This theorem is a generalization of a classical theorem which states that a geodesic is a plane curve if, and only if, it is a line of curvature, and it also provides a metric analogue of a theorem stated by Lane [4, p. 107] for projective space.

References

1. C. E. Springer, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 686–691.

2. L. P. Eisenhart, Differential Geometry, Princeton University Press, 1940.

3. C. E. Springer, Rectilinear congruences whose developables intersect a surface in its lines of curvature, Bull. Amer. Math. Soc. vol. 51 (1945) p. 992.

4. E. P. Lane, Projective differential geometry of curves and surfaces, University of Chicago Press, 1931, p. 107.

THE CURVATURES OF THE POLAR CURVES OF A GENERAL ALGEBRAIC CURVE*

EDWARD KASNER AND JOHN DE CICCO, Columbia University and Illinois Institute of Technology

1. Introduction. In this paper we present a new theorem in the polar theory of a general algebraic curve C_n of degree n. If the first polar C_{n-1} of degree $(n-1)$ with respect to the algebraic curve C_n of an ordinary point O on C_n is constructed, then C_n and C_{n-1} are initially tangent. It is found that the curvatures at O are in general distinct. We study the ratio ρ_1 of the curvature of the first polar C_{n-1} to that of C_n; it is proved that this ratio ρ_1 is given by a simple rational formula involving the degree n alone. Thus ρ_1 is independent of the nature of the algebraic curve C_n.

In particular, it follows from our theory that the polar conic C_2 of any ordinary point O of a cubic curve C_3 not only touches C_3 but also has its curvature equal to one half of the curvature of C_3 at O.

Since the rth polar C_{n-r} of degree $(n-r)$ of a point with respect to the algebraic curve C_n may be defined by induction as the first polar of the same point with respect to the $(r-1)$ polar C_{n-r+1} of degree $(n-r+1)$, our discussion is extended quite readily to the case of the rth polar curve C_{n-r} of an ordinary point O on C_n. The two algebraic curves C_n and C_{n-r} are tangent at O. We show that the ratio ρ_r of the curvature of the rth polar C_{n-r} to that of C_n is given by a simple rational formula involving the positive integers n and r only. Therefore ρ_r is independent of the polynomial equation defining C_n.

Although our new theorem is stated in metric terms, it is essentially a theorem of differential projective geometry. This is a consequence of the theorem of Mehmke-Segre which states that if any two curves are tangent at a given point O, then the ratio ρ of their curvatures at O is a projective invariant.

2. The polar curves C_{n-r} of an algebraic curve C_n. If $(x)=(x_1, x_2, x_3)$ denote homogeneous coördinates of a point in the plane, then an algebraic curve C_n of degree n is defined by the equation

$$C_n: a_x^n = f(x) = f(x_1, x_2, x_3) = \sum_{i+j+k=n} a_{ijk} x_1^i x_2^j x_3^k = 0, \tag{1}$$

where the summation extends over the indices (i, j, k) such that $i+j+k=n$.

The first polar curve C_{n-1} of degree $n-1$ of the point $(y)=(y_1, y_2, y_3)$ with respect to the algebraic curve C_n is defined by the equation [1]

$$C_{n-1}: a_y a_x^{n-1} = \Delta_y f(x) = \sum_i y_i \frac{\partial f}{\partial x_i} = y_1 \frac{\partial f}{\partial x_1} + y_2 \frac{\partial f}{\partial x_2} + y_3 \frac{\partial f}{\partial x_3} = 0. \tag{2}$$

By induction, the rth polar curve C_{n-r} of degree $n-r$ of a point with respect to the algebraic curve C_n may be defined as the first polar of the same point with respect to the $(r-1)$ polar C_{n-r+1} of degree $n-r+1$. The rth polar curve

* Presented to the American Mathematical Society, February, 1947.

C_{n-r} is given by the equation

$$C_{n-r}: a_y^r a_x^{n-r} = \Delta_y^r f(x) = \Delta_y(\Delta_y^{r-1} f(x))$$

$$= \sum_{i+j+k=r} \frac{\lfloor r}{\lfloor i \lfloor j \lfloor k} y_1^i y_2^j y_3^k \frac{\partial^r f}{\partial x_1^i \partial x_2^j \partial x_3^k} = 0. \tag{3}$$

For the operator $\Delta_y^r f(x)$ defined by equations (2) and (3), the identity

$$\frac{1}{\lfloor r} \Delta_y^r f(x) = \frac{1}{\lfloor n-r} \Delta_x^{n-r} f(y) \tag{4}$$

may be proved by means of the Taylor series expansion of $f(ax+by) = f(ax_1+by_1, ax_2+by_2, ax_3+by_3)$.

From relations (3) and (4), it is seen that the rth polar curve C_{n-r} may be defined by either of the equations

$$C_{n-r}: a_y^r a_x^{n-r} = \Delta_y^r f(x) = 0, \quad \text{or} \quad a_x^{n-r} a_y^r = \Delta_x^{n-r} f(y) = 0. \tag{5}$$

In geometric language, this means that if (x) is on the rth polar C_{n-r} of (y) with respect to the algebraic curve C_n, then (y) is on the $(n-r)$th polar C_r of (x) with respect to C_n.

From (2), it is evident that the first polar passes through all the singular points of the algebraic curve C_n. Hence any polar passes through all the singular points of the preceding polar.

3. The non-homogeneous equations of the polars C_{n-r}. In order to give a simple demonstration of our theorem, it is found advisable to obtain the equations of the polar curves C_{n-r} in non-homogeneous coördinates [2].

There is no loss in generality by assuming that (x, y) are cartesian coördinates of a point where $x = x_1/x_3$ and $y = x_2/x_3$. The equation of the algebraic curve C_n as given in the homogeneous form (1) is now of the form

$$C_n: \phi(x, y) = x_3^{-n} f(x_1, x_2, x_3) = f(x, y, 1) = 0. \tag{6}$$

Of course, $\phi(x, y)$ is a general non-homogeneous polynomial of degree n.

From (6), we may obtain the partial derivatives of first order of $f(x) = f(x_1, x_2, x_3)$ with respect to (x_1, x_2, x_3) in terms of $\phi(x, y)$ and its partial derivatives of first order with respect to (x, y). The appropriate relations are

$$\frac{\partial f}{\partial x_1} = x_3^{n-1} \phi_x, \quad \frac{\partial f}{\partial x_2} = x_3^{n-1} \phi_y, \quad \frac{\partial f}{\partial x_3} = x_3^{n-1}(n\phi - x\phi_x - y\phi_y). \tag{7}$$

Upon placing $X = y_1/y_3$ and $Y = y_2/y_3$ and using (7), we find that the first polar C_{n-1} of (X, Y) with respect to the algebraic curve C_n is

$$C_{n-1}: n\phi - (x - X)\phi_x - (y - Y)\phi_y = 0. \tag{8}$$

This is also the polar line C_1 of (x, y) with respect to C_n.

By finding the first polar of (X, Y) with respect to the algebraic curve (8), we find that the second polar C_{n-2} of (X, Y) with respect to the algebraic curve C_n is

$$C_{n-2}: n(n-1)\phi - 2(n-1)[(x-X)\phi_x + (y-Y)\phi_y] + [(x-X)^2\phi_{xx} + 2(x-X)(y-Y)\phi_{xy} + (y-Y)^2\phi_{yy}] = 0. \tag{9}$$

This is also the polar conic of (x, y) with respect to C_n.

By repeating the above process, we see that the rth polar C_{n-r} of (X, Y) with respect to C_n is

$$\begin{aligned} C_{n-r}: {}^nP_r\phi - {}^rC_1^{n-1}P_{r-1}\left[(x-X)\frac{\partial}{\partial x} + (y-Y)\frac{\partial}{\partial y}\right]^{(1)}\phi \\ + {}^rC_2^{n-2}P_{r-2}\left[(x-X)\frac{\partial}{\partial x} + (y-Y)\frac{\partial}{\partial y}\right]^{(2)}\phi - \cdots \\ + (-1)^k {}^rC_k^{n-k}P_{r-k}\left[(x-X)\frac{\partial}{\partial x} + (y-Y)\frac{\partial}{\partial y}\right]^{(k)}\phi + \cdots \\ + (-1)^r\left[(x-X)\frac{\partial}{\partial x} + (y-Y)\frac{\partial}{\partial y}\right]^{(r)}\phi = 0. \end{aligned} \tag{10}$$

This is also the $(n-r)$ polar C_r of (x, y) with respect to C_n.

4. The ratio ρ_1 of the curvatures of C_{n-1} and C_n. The slope $dy/dx = y'$ at any point (x, y) of C_{n-1} as given by (8), is determined by the equation

$$(n-1)(\phi_x + y'\phi_y) - (x-X)(\phi_{xx} + y'\phi_{xy}) - (y-Y)(\phi_{xy} + y'\phi_{yy}) = 0. \tag{11}$$

If $O(X, Y)$ is an ordinary point of the algebraic curve C_n, then from (8), it is seen that by taking $x = X$ and $y = Y$, the point $O(X, Y)$ is on the first polar curve C_{n-1}. By (11), it is found that the slope y' of C_{n-1} at O is $-\phi_x/\phi_y$, which is also the slope Y' of C_n at O. Therefore C_n and C_{n-1} are tangent at O. Hence all the polar curves of an ordinary point O of C_n with respect to C_n are tangent to C_n to O. From this follows the well-known result that the polar line of an ordinary point O of an algebraic curve C_n is the tangent line of C_n at O.

Now we are in a position to state and prove our new proposition.

THEOREM. *The ratio ρ_1 of the curvature of the first polar C_{n-1} to that of the algebraic curve C_n, constructed at an ordinary point $O(X, Y)$ of C_n, is*

$$\rho_1 = \frac{n-2}{n-1}. \tag{12}$$

For example, the curvature of the polar conic section C_2 with respect to a cubic curve C_3 of an ordinary point O of C_3, is $1/2$ the curvature of C_3.

As another example, the curvature of the polar cubic C_3 with respect to a quartic curve C_4 at an ordinary point O of C_4 is $2/3$ that of C_4.

To prove our theorem, we proceed in the following manner. The second derivative $d^2y/dx^2=y''$ at any point (x, y) of the first polar C_{n-1} is found by differentiating the equation (11) with respect to x. The result is

$$\begin{aligned}(n-2)(\phi_{xx}+2y'\phi_{xy}+y'^2\phi_{yy})+(n-1)\phi_y y''\\ -(x-X)(\phi_{xxx}+2y'\phi_{xxy}+y'^2\phi_{xyy}+y''\phi_{xy})\\ -(y-Y)(\phi_{xxy}+2y'\phi_{xyy}+y'^2\phi_{yyy}+y''\phi_{yy})=0.\end{aligned} \tag{13}$$

At the point O where $x=X$, $y=Y$, the second derivative y'' of C_{n-1} is given by the equation

$$(n-2)(\phi_{xx}+2y'\phi_{xy}+y'^2\phi_{yy})+(n-1)\phi_y y''=0. \tag{14}$$

We recall that the second derivative Y'' of the algebraic curve $C_n:\phi(x, y)=0$, is determined by the equation

$$\phi_{xx}+2Y'\phi_{xy}+Y'^2\phi_{yy}+\phi_y Y''=0. \tag{15}$$

Hence using the fact that $y'=Y'$ at the point 0 where $x=X$, $y=Y$, we deduce from the equations (14) and (15), the following result

$$(n-1)y''=(n-2)Y''. \tag{16}$$

Finally since $y'=Y'$, the ratio ρ_1 of the curvature of the first polar curve C_{n-1} to that of C_n at the point $O(X, Y)$ is

$$\rho_1=\frac{y''}{Y''}=\frac{n-2}{n-1}. \tag{17}$$

This completes the proof of our theorem.

5. The polar curves C_{n-r} of the origin. Before giving an alternate proof of our theorem by the application of infinite series expansions, we shall develop the equations of the various polar curves C_{n-r} of the origin $O(0, 0)$ (which need not be on C_n) with respect to the algebraic curve C_n.

By (1) and (6), any algebraic curve C_n of degree n can be written in the form

$$C_n:\phi(x, y)=\sum_{k=0}^{n}P_k(x, y)=0, \tag{18}$$

where $P_k(x, y)$ is a homogeneous polynomial in (x, y) of degree k.

By (8), it is found that the first polar C_{n-1} of the origin O with respect to C_n is

$$C_{n-1}:\sum_{k=0}^{n-1}(n-k)P_k(x, y)=0. \tag{19}$$

Upon finding the first polar of the origin with respect to this curve (19), we find that the second polar C_{n-2} of the origin with respect to C_n is

$$C_{n-2}:\sum_{k=0}^{n-2}(n-k)(n-k-1)P_k(x, y)=0. \tag{20}$$

By repeated application of the above process, we find that the rth polar curve C_{n-r} of the origin with respect to C_n is

$$C_{n-r}: \sum_{k=0}^{n-r} (n-k)(n-k-1)\cdots(n-k-r+1)P_k(x, y) = 0. \tag{21}$$

Upon placing $r=n-2$ in the preceding equation, the polar conic C_2 of the origin is

$$C_2: n(n-1)P_0 + (n-1)P_1(x, y) + P_2(x, y) = 0. \tag{22}$$

The polar line C_1 is

$$C_1: nP_0 + P_1(x, y) = 0. \tag{23}$$

By (18) and (23), it is deduced that the polar line C_1 may be defined geometrically as follows. Draw any line L through the point O. This line L intersects the algebraic curve C_n in the n points $(P_1, P_2, \cdots, P_n)$ which may be coincident or imaginary. Let P be the generalized harmonic mean of these n points; that is, let P be such that $1/OP = (1/n)\sum_{i=1}^{n} 1/(OP_i)$. As L rotates about the point O, the point P describes the polar line C_1 of O.

Having defined the polar line C_1 of the point O geometrically, the first polar C_{n-1} of the point O is seen by (8) to be the locus of points whose polar lines pass through the point O. By induction, the rth polar C_{n-r} of the point O is the locus of points whose polar lines with respect to the $(r-1)$ polar C_{n-r+1} pass through the point O.

If C_{n-r} is the rth polar of a point O with respect to C_n, then on any line L passing through O, the generalized harmonic mean of the n intersections of L with C_n coincides with that of the $(n-r)$ intersections of L with C_{n-r} [3].

6. Alternate proof of the theorem. Let O be an ordinary point of the algebraic curve C_n. Then by (18), $P_0=0$ and the equation of the tangent line of C_n at O is $P_1(x, y)=0$. By the formulas of the preceding section, it is verified that all the polar curves of O with respect to C_n are tangent to C_n at O.

There is no loss of generality if we assume $P_1(x, y) = -y$. Let ϕ_{xx} have the value $2a_{20}$ at the origin. By equations (18) and (19), we find that the expressions for y as power series in x of the algebraic curve C_n and the first polar curve C_{n-1} are

$$C_n: y = a_{20}x^2 + \cdots; \quad C_{n-1}: y = \frac{n-2}{n-1}a_{20}x^2 + \cdots. \tag{24}$$

Therefore at the origin O, the ratio ρ_1 of the curvature of C_{n-1} to that of C_n is

$$\rho_1 = \frac{\dfrac{n-2}{n-1}a_{20}}{a_{20}} = \frac{n-2}{n-1}. \tag{25}$$

7. The ratio of the curvature of the rth polar curve C_{n-r} to that of C_n. By using the fact that the rth polar curve C_{n-r} of a point O with respect to the algebraic curve C_n is the first polar curve of O with respect to the $(r-1)$ polar curve C_{n-r+1} of O with respect to C_n, we can extend our theorem in the following way.

THEOREM. *The ratio ρ_r of the curvature of the rth polar curve C_{n-r} to that of the algebraic curve C_n, constructed at an ordinary point O of C_n, is*

$$\rho_r = \frac{n-r-1}{n-1}. \tag{26}$$

For example, consider a general quintic curve C_5. At an ordinary point O of C_5, construct the polar quartic C_4, the polar cubic C_3, the polar conic C_2, and the polar line C_1. The corresponding ratios of the curvatures are 3/4, 1/2, 1/4, and 0.

Another proof of this theorem is the following one. In equation (18), $P_0=0$, and we can take $P_1(x, y) = -y$. Let ϕ_{xx} have the value $2a_{20}$ at the origin. By (18) and (21), the expressions for y as power series in x of the algebraic curve C_n and the rth polar curve C_{n-r}, are

$$C_n: y = a_{20}x^2 + \cdots, \qquad C_{n-r}: y = \frac{n-r-1}{n-1} a_{20}x^2 + \cdots. \tag{27}$$

Hence at the origin O, the ratio ρ_r of the curvature of C_{n-r} to that of C_n is

$$\rho_r = \frac{\dfrac{n-r-1}{n-1} a_{20}}{a_{20}} = \frac{n-r-1}{n-1}. \tag{28}$$

8. Concluding remarks. We have also studied the ratio ρ_r of the departure of the polar curve C_{n-r} to that of the algebraic curve C_n from their common tangent line in the case where the order of contact is p. We have found that this ratio ρ_r is given by a simple rational formula involving only the positive integers (n, r, p). In particular, this applies to inflections.

Also we have given some consideration to the case where the point O on C_n is a singularity. Both C_n and C_{n-r} have a singularity of the same qualitative nature at O. An analogous ratio ρ_r can be constructed. We find that in many cases, this ratio ρ_r depends on the coefficients of the polynomial defining the algebraic curve C_n.

REFERENCES

1. Snyder and Sisam, Analytic geometry of space, Henry Holt and Company, New York (1928).

2. Kasner, Determination of the algebraic curves whose polar conics are parabolas, Amer. Journ. Math. 26, 164–168 (1903).

3. Salmon, Treatise on higher plane curves, Dublin: Hodges and Smith, Grafton Street (1852).

THE DETERMINATION OF THE COMPLEX ZEROS OF A POLYNOMIAL

G. C. BEST, San Diego, California

1. Introduction. In this paper a method is explained for the determination of the real parts of complex zeros of a polynomial once their moduli are known. The method discussed is intended for use with Graeffe's method [1, 2, 3] of root squaring, and could very easily be treated in a class in the Theory of Equations after Graeffe's method for the real roots and the moduli of complex roots of real polynomials had been considered. Graeffe's method, though very convenient when real roots alone are involved, begins to become awkward when complex roots appear, due to the difficulty in determining the arguments of such roots. The difficulty arises largely because of the lack of a method which can be extended easily to cover cases involving many complex roots. The following procedures are offered in the hope of reducing some of these difficulties.

2. The general method. Let us consider the polynomial

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \cdots + a_nx^n = 0,$$

in which $a_0 \neq 0$. Moreover, let us assume that the modulus r of a pair of complex roots is known. Let $r^2=\beta$. Then we consider the factor $x^2+\alpha x+\beta$, where $-\alpha/2$ equals the real part of either complex root in question. Let $u=-\alpha/\beta$ and $k=-1/\beta$, and expand the fraction $P(x)/(1-ux-kx^2)$ in ascending powers of x. The coefficient of x^r, $r=1, 2, \cdots$, in this expansion is a polynomial $p_r(u)$ of degree r in u. A study of the process of synthetic division applied to the above fraction shows that the p_r's are connected by the relationship:

$$\begin{aligned} p_0 &= a_0 \\ p_1 &= a_0u + a_1 \\ &\cdots\cdots\cdots \\ p_r &= up_{r-1} + kp_{r-2} + a_r \qquad (r = 2, 3, \cdots). \end{aligned} \tag{1}$$

THEOREM. *A necessary and sufficient condition that the coefficients of x^n and x^{n-1} be zero is that $1-ux-kx^2$ or, what amounts to the same thing, $\beta+\alpha x+x^2$ is a factor of $P(x)$.*

To show that this condition is necessary, consider the coefficients of x^n and x^{n-1} zero. Then by relation (1), all coefficients p_r $(r>n)$ also equal zero, it being understood that $a_r=0$ for $r>n$, that is, there is no remainder. Therefore, $\beta+\alpha x+x^2$ is a factor.

To show that the condition is sufficient, assume that $\beta+\alpha x+x^2$ is a factor of $P(x)$. In this case the quotient will be a polynomial of degree $(n-2)$, and there will be no remainder; hence all coefficients p_r for $r>(n-2)$ will be zero, in particular, those of x^n and x^{n-1}.

The greatest common divisor of p_n and p_{n-1} is then obtained by the usual

process. This common factor, set equal to zero, will then give the desired value of u, for the argument of the complex number involved is easily determined from u.

If, when determining the zeros of $P(x)$ by Graeffe's method, it is found that s pairs of complex roots have the same modulus r, then the greatest common divisor of p_n and p_{n-1} will be a polynomial of degree s, and it will have real zeros. The determination of these zeros can be made by Graeffe's method if necessary. If the formulas for the determination of α shown in Table I are employed, consideration must be given to multiple moduli, for if an s-fold modulus exists, the formulas for $(s-1)$-fold or less will give the result 0/0.

TABLE I. FORMULAS FOR α FOR KNOWN β

Degree n of $P(x)$	Number of identical moduli	Formula from which to determine α in factor $x^2+\alpha x+\beta$
3	1	$\alpha=\dfrac{a_1\beta-a_3\beta^2}{a_0}$
4	1	$\alpha=\dfrac{a_1\beta-a_3\beta^2}{a_0-a_4\beta^2}$
	2	$a_0\alpha^2-(a_1\beta)\alpha-(a_0\beta-a_2\beta^2+a_4\beta^3)=0$
5	1	$\alpha=\dfrac{a_0a_1\beta-a_0a_3\beta^2+a_2a_4\beta^4-a_4a_5\beta^5}{+a_0-a_0a_4\beta^2+a_1a_5\beta^3-a_5^2\beta^5}$
	2	$a_0\alpha^2-(a_1\beta-a_5\beta^3)\alpha-(a_0\beta-a_3\beta^2+a_4\beta^3)=0$

3. The derivation of formulas for $n=3, 4, 5$. Consider the first division in the usual algorithm for the determination of the greatest common divisor of $p_n(u)$ and $p_{n-1}(u)$. Since $p_n(u)=up_{n-1}(u)+kp_{n-2}(u)+a_n$, evidently $kp_{n-2}(u)+a_n$ is the first remainder. (The first quotient is u.) Any common factor of p_n and p_{n-1} is a factor of

$$kp_{n-2}(u)+a_n. \tag{2}$$

If $n=3$, the polynomial (2) is linear, and must give the common root u that is desired. This gives

$$\alpha=\frac{\beta a_1-\beta^2a_3}{a_0}.$$

The next step in the greatest common divisor algorithm, unnecessary when $n=3$, calls for dividing $p_{n-1}(u)$ by the polynomial (2). Since

$$p_{n-1}(u)=up_{n-2}(u)+kp_{n-3}(u)+a_{n-1},$$

we have

$$kp_{n-1}(u)-u[kp_{n-2}(u)+a_n]=k^2p_{n-3}(u)+ka_{n-1}-a_nu, \tag{3}$$

where the dividend $p_{n-1}(u)$ has been multiplied by the constant k, a procedure that is permissible in the algorithm. The right member of (3) is the desired linear

common factor when $n=4$, $s=1$, and is a polynomial of degree $(n-3)$, divisible by the greatest common divisor of p_n and p_{n-1} when $n=5$. The next step in the algorithm results in

$$k^3p_{n-4}(u) + k^2a_{n-2} + ka_n - ka_{n-1}u + a_nu^2. \tag{4}$$

The process will not be carried farther as the resulting formulas become impracticably complicated. Equations (1) to (4) yield the results shown in Table I for polynomials up to the fifth degree. The p_r's can be developed from equation (1) with the substitutions $u=-\alpha/\beta$ and $k=-1/\beta$ made to obtain the final forms indicated.

When determining the arguments for complex zeros of polynomials of degree higher than the fifth, it is suggested that each time a pair of complex zeros is completely determined $P(x)$ be divided by the associated quadratic factor, thus reducing its degree and the amount of labor involved in determining the remaining zeros. When the resulting polynomial is eventually reduced to one of fourth or fifth degree, use can be made of the equations of Table I, thereby greatly facilitating the completion of the work.

4. Tabular method. As an illustrative example, consider the polynomial

$$x^5 + 8x^4 + 9x^3 - 22x^2 - 36x - 72 = 0.$$

Assume it is known that the moduli of one pair of complex zeros of this polynomial have the product 2. Then $\beta=2$ and $k=-1/2$.

A tabular method based upon equation (1) is illustrated in Table II. The coefficients of $P(x)$ are arranged in ascending powers of x across the top of the table. In column 3 the degree r of the resulting polynomial in u is indicated. The coefficients for the polynomials in u are arranged in horizontal rows for descending powers of u. For example, for $r=3$, reference to row 6 reveals the polynomial

$$-72u^3 - 36u^2 + 50u + 27.$$

This polynomial is the coefficient of x^3 in the expansion of

$$\frac{x^5 + 8x^4 + 9x^3 - 22x^2 - 36x - 72}{1 - ux + .5x^2}.$$

Mention should be made of the shift of columns horizontally in connection with the explanatory equations shown in column 2. For instance, the elements of row 3, designated as R_3, are equal to kp_1 or $-\frac{1}{2}R_1$, moved two columns to the right. A similar shift is involved in R_5, R_7, and R_9. Each number in a row for p_r is the sum of the two numbers above it. The coefficients a_r across the top line are added to the elements in the corresponding lines p_r. The elements of row 10 correspond to p_5; at that point the procedure for building p_r's is stopped. The process thereafter is that of determining the greatest common divisor of p_4 and p_5. Row 11, for example, is $-up_4$ so as to match the highest

powers of u. Row 12 $=p_5-up_4$. Subsequent processes are similar, namely, that of shifting columns to match highest powers of u and multiplying by constants to equate corresponding coefficients, continually reducing the degree of the difference. In row 21 the equation $2u+1=0$ is indicated. Hence $u=-1/2$, and since $u=-\alpha/\beta$,

$$\alpha = -\beta u = -2(-\tfrac{1}{2}) = +1.$$

Therefore the quadratic factor sought is x^2+x+2, and the complex roots finally obtained are

$$x = \frac{-1 \pm \sqrt{7}\, i}{2}.$$

TABLE II. COMPUTATION OF u FOR KNOWN k

1	2	3	4 a_0	5 a_1	6 a_2	7 a_3	8 a_4	9 a_5	10	11
Item	Equation	p_r	−72	−36	−22	+ 9	+ 8	+ 1		
1		p_0	0 −72							
2		p_1	0 −72	0 −36						
3	$k\times R_1^*$		0	0	+36					
4	R_2+R_3	p_2	−72	−36	+14					
5	$k\times R_2^*$		0	0	+36	+18				
6	R_4+R_5	p_3	−72	−36	+50	+27				
7	$k\times R_4^*$		0	0	+36	+18	− 7			
8	R_6+R_7	p_4	−72	−36	+86	+45	+ 1			
9	$k\times R_6^*$		0	0	+ 36	+18	−25	−13.5		
10	R_8+R_9	p_5	−72	−36	+122	+63	−24	−12.5		
11	$-R_8$		+72	+36	−86	−45	− 1	0		
12	$R_{10}+R_{11}$				+36	+18	−25	−12.5		
13	$2\times R_{12}$				+72	+36	−50	−25	0	
14	$-R_{11}$				−72	−36	+86	+45	+ 1	
15	$R_{13}+R_{14}$						+36	+20	+ 1	
16	$-R_{12}$						−36	−18	+ 25	+ 12.5
17	$R_{15}+R_{16}$							+ 2	+ 26	+ 12.5
18	$18\times R_{17}$							+36	+468	+225
19	$-R_{15}$							−36	− 20	− 1
20	$R_{18}+R_{19}$								+448	+224
21	$R_{20}/224$								2	1

* This result is to be moved two rows to the right.

The method, though occupying considerable space, is fairly simple to follow and proceeds rather rapidly. If the zeros are required to many decimal places, a vertical arrangement would be superior. Also if a calculating machine is used, the operations corresponding to rows 3, 5, 7, 9, and so on, in Table II may be made upon the machine and omitted from the table.

If β is known only approximately, the assumption that the coefficients of x^{n-1} and all higher powers of x are zero is slightly in error. Consequently, a precise common factor will not exist; since, however, the degree of the desired common factor is known, that is, the multiplicity of the modulus, there is no ambiguity as to when to stop the greatest common divisor algorithm. The resulting α will, in such cases, also be approximate.

References

1. E. T. Whittaker and G. Robinson, The Calculus of Observations, Blackie & Son, 2nd ed. p. 106.

2. C. Runge and H. Konig, Numerisches Rechnen, Julius Springer, p. 164.

3. Th. v. Kármán and T. A. Biot, Mathematical Methods in Engineering, McGraw-Hill, 1st ed. p. 194.

MATHEMATICAL NOTES

EDITED BY E. F. BECKENBACH, University of California

Material for this department should be sent directly to E. F. Beckenbach, University of California, Los Angeles 24, California.

A SLOWLY DIVERGENT SERIES

R. P. AGNEW, Cornell University

It is well known, and is easily proved by the integral test for convergence and divergence of series, that the series

$$(1) \qquad \sum \frac{1}{n}, \quad \sum \frac{1}{n \log n}, \quad \sum \frac{1}{n \log n \log \log n}, \ \ldots$$

are all divergent. These are classic examples of slowly divergent series, each one after the first diverging more slowly than its predecessor. The following theorem, which is a simple corollary of a theorem of B. Pettineo,* gives a series which diverges more slowly than any of those in (1).

* B. Pettineo. Estensione di una classe di serie divergenti. Atti della Reale Accademia Nazionale dei Lincei, Rendiconti, Classe di Scienze Fisiche, Matematiche e Naturali. Series 8, vol 1 (1946), pp. 680–685.

THEOREM. *For each $x>0$, let $P(x)$ denote the product of x and all of the numbers*

$$\log x, \quad \log\log x, \quad \log\log\log x, \cdots \tag{2}$$

which are greater than 1. *Then the series*

$$\sum_{n=1}^{\infty} \frac{1}{P(n)} \tag{3}$$

is divergent.

Following a method different from that of Pettineo we shall use the integral test, proving that (3) is divergent by proving that

$$\int_1^{\infty} \frac{1}{P(x)}\,dx = \infty.$$

The function $1/P(x)$ is continuous and decreasing over the infinite interval $x \geqq 1$. Over the interval $1 \leqq x \leqq e$, $1/P(x)$ is $1/x$; hence

$$\int_1^{e} \frac{1}{P(x)}\,dx = \log x \Big]_1^{e} = 1.$$

Over the interval $e \leqq x \leqq e_2 = e^e$, $1/P(x)$ is $1/(x \log x)$; hence

$$\int_e^{e_2} \frac{1}{P(x)}\,dx = \log\log x \Big]_e^{e_2} = 1.$$

Over the interval $e_2 \leqq x \leqq e_3 = \exp e_2$, $1/P(x)$ is $1/[x(\log x)\log\log x]$; hence

$$\int_{e_2}^{e_3} \frac{1}{P(x)}\,dx = \log\log\log x \Big]_{e_2}^{e_3} = 1.$$

Continuation of this process gives an infinite set of intervals (of rapidly increasing lengths) over each of which the integral of $1/P(x)$ is 1. The results follow from this.

RESIDUE OF $\sigma_k(n)$ MODULO 2

A. R. NASIR, Talim-ul-Islam College, India

Let $\sigma_k(n) = \sum_{d/n} d^k$, so that $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n. By entirely elementary methods we prove two theorems which completely determine the residue of $\sigma_k(n)$.

THEOREM A. *If $n = 2^a \prod_{p/n} p^{2b_i}$, $a \geqq 0$, $b_i \geqq 0$ for every i, $1 \leqq i \leqq m$, and p's are distinct primes >2, then $\sigma_k(n) \equiv 1 \pmod 2$.*

It is sufficient to consider the case when $a>0$ and $b_i>0$ for every i. Write

$$\sigma_k(n) = \left(1 + \sum_{x=1}^{x=a} 2^{xk}\right) \prod_{i=1}^{i=m} \left(1 + \sum_{y=1}^{y=2b_i} p^{yk}\right).$$

Now $\sum_{\nu=1}^{\nu=3b_i} p^{\nu k}$, for $i=j$, represents the sum of an even number of integers each $\equiv 1 \pmod 2$.

Therefore the factor of $\sigma_k(n)$ corresponding to any p is congruent to 1 (mod 2). The factor corresponding to 2 evidently is congruent to 1 (mod 2). Hence our result follows.

THEOREM B. *If* $n=2^a p^b \prod_{q/n} q^{c_i}$, $a \geqq 0$, b *an odd integer* >0, $c_i \geqq 0$ *for every* i, $1 \leqq i \leqq m$, p *and* q*'s all distinct primes* >2, *then* $\sigma_k(n) \equiv 0$ (*mod* 2).

Here the factor corresponding to p is congruent to 0 (mod 2). For, $\sum_{\nu=1}^{b_i} p^{bk}$ represents the sum of an odd number of integers each $\equiv 1 \pmod 2$.

NAGEL POINT IN THE TETRAHEDRON

V. THÉBAULT, Tennie, Sarthe, France

1. A well known property. In a triangle ABC, the lines joining the vertices A, B, C to the contact points D_a, E_b, F_c of the ex-circles I_a, I_b, I_c with the sides BC, CA, AB, respectively, between B and C, C and A, A and B, meet at the Nagel point N of the triangle; and the points D_a, E_b, F_c divide the perimeter of the triangle in equivalent parts

$$\begin{aligned} AB + BD_a &= D_aC + CA = BC + CE_b = E_bA + AB \\ &= CA + AF_c = F_cB + BC \\ &= \frac{BC + CA + AB}{2}, \end{aligned}$$

and conversely. Further, the Nagel point is the in-center of the anticomplementary triangle $A_1B_1C_1$, and its associates N_a, N_b, N_c have similar properties.

2. Analogue in space. Let $T \equiv ABCD$ be any tetrahedron, a and a', b and b', c and c' the lengths of the edges BC and DA, CA and DB, AB and DC; A, B, C, D the areas of the faces BCD, CDA, DAB, ABC; h_a, h_b, h_c, h_d the altitudes from A, B, C, D; r the radius of the in-sphere; V the volume. A point A' in the plane BCD is such that

$$(1) \qquad D + BCA' = B + CDA' = C + DBA' = \frac{(A + B + C + D)}{3},$$

if and only if

$$D + \frac{ax}{2} = B + \frac{c'y}{2} = C + \frac{b'z}{2} = \frac{A + B + C + D}{3},$$

x, y, z being the distances from A' to the edges BC, CD, DB. It follows that

$$ax = \frac{A + B + C + D - 3D}{3} = 2V\left(\frac{1}{2} - \frac{3}{h_d}\right) = \frac{2A(h_d - 3r)}{3r};$$

by circular permutations we find similar values for $c'y$, $b'z$; and we have also similar relations for the points B', C', D' in the planes CDA, DAB, ABC.

These barycentric coördinates $(ax, c'y, b'z), \cdots$, of the points A', B', C', D', in the triangles BCD, CDA, DAB, ABC, are proportional to the barycentric coördinates $A(h_a-3r)$, $B(h_b-3r)$, $C(h_c-3r)$, $D(h_d-3r)$ with respect to T, of the center I_1, of the in-sphere of the tetrahedron $T_1 \equiv A_1B_1C_1D_1$, anticomplementary to T, and whose distances to the faces of T are h_a-3r, h_b-3r, h_c-3r, h_d-3r. The lines AI_1, BI_1, CI_1, DI_1 meet the faces BCD, CDA, DAB, ABC, of T at the points A', B', C', D' corresponding to the above relations (1) and to the other similar ones. Hence we have the following theorem.

THEOREM. *In any tetrahedron $T \equiv ABCD$, the lines AA', BB', CC', DD' joining the vertices to the points A', B', C', D', corresponding to the relations* (1) *and to the other similar ones, meet at a point (Nagel point) which is also the center, I_1, of the in-sphere of the anticomplementary tetrahedron T_1.*

To the point I_1 correspond *seven associated points* coinciding with the centers of the four escribed spheres contained in the trunks and the three escribed spheres contained in the roofs. These seven points correspond to the sets of points (A'', B'', C'', D'') and (A''', B''', C''', D''') in the planes BCD, CDA, DAB, ABC. Thus we have the relations between areas,

$$-D + BCA'' = -B + CDA'' = -C + DBA'' = \frac{A - B - C - D}{3}, \cdots$$

similar to (1), corresponding to the center of the escribed sphere of T_1 contained in the trunk opposite to the vertex A_1.

It is to be noted that, generally, the lines AA', BB', CC', DD' do not pass through the contact points of the escribed spheres with the faces of the tetrahedron T.

CLASSROOM NOTES

EDITED BY C. B. ALLENDOERFER, Haverford College

All material for this department should be sent to C. B. Allendoerfer, Haverford College, Haverford, Pennsylvania. Contributions are invited on topics of immediate interest to teachers of undergraduate mathematics such as: fresh approaches to standard material, analyses of common textbook shortcomings, descriptions of visual and mechanical aids to teaching, outlines of new types of courses, and discussions of the role of mathematics in the revised curricula being adopted by many institutions. Rejoinders to earlier notes are encouraged.

FOUR USEFUL BLACKBOARD AIDS

R. M. SUTTON, Haverford College

In the teaching of physics, I am constantly aware of the enormous part played by mathematics in the development of that science. It is therefore with satisfaction that I make a partial payment of my debt to mathematics in these suggested simple, physical teaching aids.

1. A protractor and slope indicator. Next to a piece of chalk and an eraser, I find the most useful article for blackboard use is a straight edge with a plumb-bob and protractor. That may sound complicated, but it is really very simple. For the past five years I have used a homemade gadget like the one shown in Figure 1. This figure is approximately one fifth the actual size. It consists of a

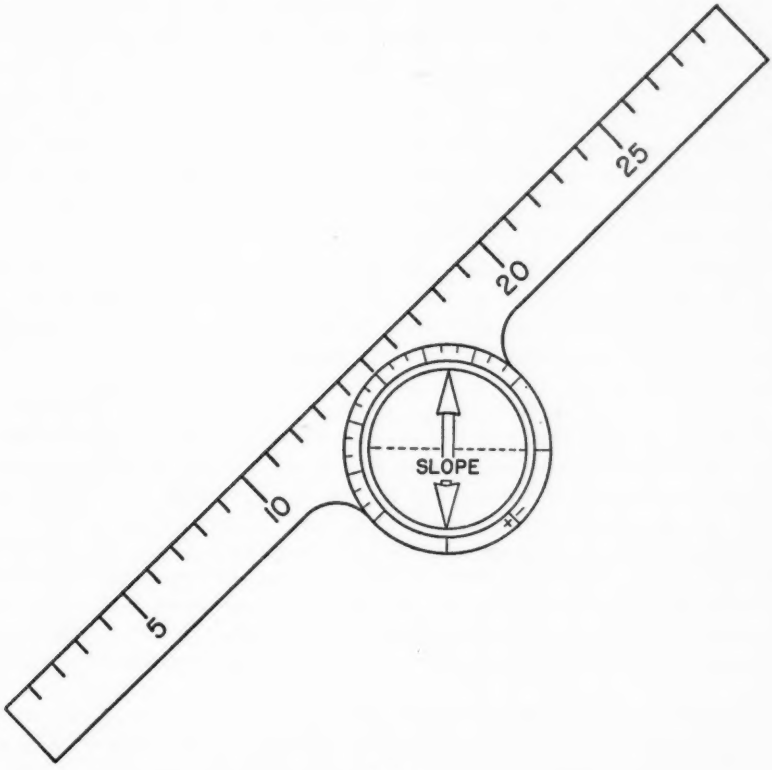

FIG. 1

straight edge about 30 inches long, equipped with a circular scale. At the center of the scale is a pivot on which turns a loaded disk whose center of gravity is well below the pivot. As the straight edge is tilted, this disk remains fixed with respect to the earth and it therefore turns relative to the zero point on the scale. A vertical arrow on the disk points to 0° on the scale when the edge is horizontal. It likewise indicates the inclination of the edge in any other orientation.

The upper half of the scale is graduated from $-90°$ to $+90°$. For analytic geometry, a second arrow pointing downward traverses another scale on the

lower half of the circle. This second scale is graduated directly in slope, positive on the left, negative on the right.

The protrusion on the side of the straight edge makes a convenient handle by which to grip the instrument while chalk is being run along the drawing edge. If the disk rotates freely, the error in setting the instrument does not exceed one degree.

Among the uses of the instrument are: drawing accurately parallel, vertical or horizontal lines, or lines at any desired inclination, drawing parallelograms, equilateral triangles, or other regular polygons in any orientation; drawing orthogonal coördinate axes in any orientation, especially good for showing transformation of coördinates; and drawing components of a vector resolved along arbitrarily chosen axes. With a few minutes practice, excellent five-pointed stars can be drawn. The ease and accuracy with which the instructor can produce accurate figures is sufficient justification for the expenditure of the few extra seconds required to set the directed straight edge on the blackboard.

2. Slides for projecting coördinate systems. In the teaching of coördinate geometry, the instructor either wastes time drawing accurate coördinate systems on which to locate specified points or lines (Instrument #1 may improve this part of his work), or he may draw a slap-dash figure that later embarrasses him and leaves the class with a low estimate of his muscular ability. A simple technique is available for obtaining a coördinate system almost instantly: use a projection lantern and an appropriate slide. Simply prepare a transparent slide for the kind of coördinate system desired and project it on the blackboard. Curtains do not have to be drawn. Like the use of black chalk on a blackboard (which sounds absurd, but is actually quite feasible), the use of a slide makes the coördinate system show up clearly. The lines of the system are not erasable, a feature which is very convenient when changes in the figure are to be made. The only disadvantage is that the system disappears from view under the shadow of the instructor's hand or body, but this may help to remind him to stand out of the way of his work whenever he wants students to see what he has drawn.

Suggested slides for use in the projection lantern are: plane cartesian coördinates with origin at center of field of view; cartesian system for first quadrant only; plane polar coördinates; semi-logarithmic coördinates; log-log coördinates; plane representation of three-dimensional cartesians or polars, and so forth. Slides may be either made photographically from systems drawn on paper, or they may be prepared by drawing with India ink on cellophane.

3. An improved compass. The simplest and least expensive blackboard compass is a string tied to a piece of chalk. For many purposes this is quite satisfactory. My previous experience with mechanical blackboard compasses leaves me with a low opinion of those that I have seen. The one described here has some advantages not offered by others, and it is recommended where precise drawings are desired.

The basic element of the compass shown in Figure 2 (approximately full size), is an ordinary 6-foot steel tape-measure which rolls into a circular holder about 2 inches in diameter. This holder is modified by the addition of a handle, pivot, and suction cup. The suction cup establishes the center on the blackboard. Moreover, the holder is provided with a narrow channel through which the tape is pulled out and pushed in. This channel offers enough friction to prevent the

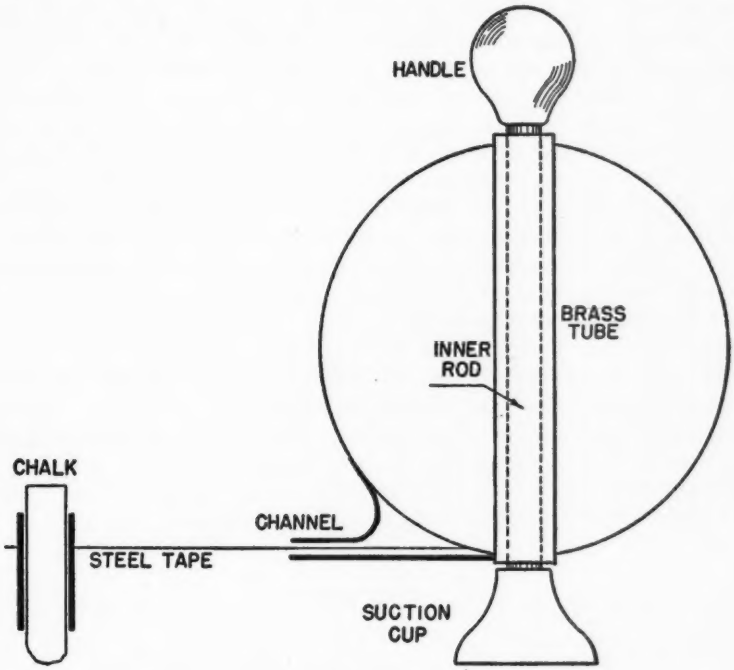

FIG. 2

tape from moving unless the operator chooses to change its length. A tubular fixture is soldered to the end of the tape to hold a piece of chalk. The radius of the circle for which the tape is set can be read immediately from a fixed reference point.

The instrument offers compactness, ease and precision of setting, convenience in drawing, and a long radius (arm's length). It is not very good for circles under five or six inches in radius but it is excellent for larger circles. For drawing a series of concentric circles of specified radii, as in diagrams pertaining to gravitational or electric potential or to spherical wave patterns, it is invaluable.

Only one precaution is needed in using the instrument. The tape may be pulled out easily, but care must be exercised in pushing it back into the holder.

It is not very rigid under compression and may be damaged by sharp bends. Therefore, the operator should push back only two or three inches at a time to prevent buckling under compression.

4. Templates for drawing sine curves. Almost too trivial to mention, yet very useful, is a set of templates cut from thick cardboard or from $\frac{1}{4}$-inch plywood for drawing sine and cosine curves. I have seen only one mechanical blackboard sine curve machine. It was complicated, expensive, and not very satisfactory. It would be desirable to have such an instrument constructed in such a manner that there would be no limitation to the possible range of amplitude of the curve and in the scale of length per cycle, but there does not seem to be any simple and satisfactory design. However, it is easy to make a set of templates, any one of which can be held against the blackboard to serve as a pattern for a portion of a sine or cosine curve. These templates may differ in amplitude and length of a cycle. I find them useful when drawing figures to accompany the discussion of simple harmonic motion and wave motions. Particularly when matters of phase difference or interference are involved, they are far superior to free-hand drawings.

Three styles will cover most needs:

(a) a full cycle of amplitude A and length l.

(b) two half cycles of different A and l; these may conveniently form the upper and lower edges of the template.

(c) a half cycle of $A \sin \theta$ and a full cycle of $A \sin^2 \theta$, again forming the upper and lower edges of the same template.

It is convenient to give each template a handle, to indicate on it the value of A and l, and to mark the ordinate at every 15 or 30 degrees.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Oregon State College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Oregon State College, Corvallis, Oregon. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 771. *Proposed by C. C. Carter, Bluffs, Illinois*

A die bearing the numbers 0, 1, 2, 3, 4, 5 on its faces is repeatedly thrown until the total of the throws first exceeds 12. What is the most likely total that will be thus obtained?

E 772. *Proposed by D. H. Browne, Buffalo, N. Y.*

What is the number of shortest paths between two points in an n-dimensional lattice?

E 773. *Proposed by A. L. Rubinoff, University of Toronto*

Suppose that noughts and crosses are played on an n-dimensional cube of side k. Show that there are precisely

$$\frac{(k+2)^n - k^n}{2}$$

rows, columns, diagonals, $\cdots$ on which a win may be scored.

E 774. *Proposed by Norman Anning, Ann Arbor, Michigan*

Consider points on the median of a triangle. Through the centroid no straight line can be drawn which will cut off one-third of the area. Through a point four-fifths of the distance from vertex to base, four such lines can be drawn. Find points on the median at which the number of possible lines changes.

E 775. *Proposed by R. P. Boas, Jr., Brown University*

Consider a determinant of order n whose elements are x on the main diagonal, ± 1 elsewhere. Find the smallest positive number a such that for $x > a$ the determinant is positive for all choices of the $\pm$ signs.

SOLUTIONS

Cupid's Problem

E 740 [1946, 462]. *Proposed by Esther Szekeres, Shanghai, China*

Let there be given five points in the plane. Prove that we can select four of them which determine a convex quadrilateral.

I. *Solution by P. T. Bateman, Yale University.* Let W be the boundary of the convex hull of the five points. If all five points lie on W, then either all five points are collinear or else any non-collinear set of four of the points has the desired property. If exactly four of the given points lie on W, then these four points have the desired property.

If exactly three of the given points A, B, C lie on W, then A, B, C are the vertices of a triangle enclosing the other two points D and E. Now the line DE cuts the interior of at most two sides of triangle ABC. Consider a side whose interior DE does not cut. Then D, E, and the two vertices on this side have the desired property.

The general case of this problem is discussed by Erdös and G. Szekeres in an article in *Compositio Mathematica*, vol. 2 (1935), pp. 463–470. From nine

points in the plane we can select five which determine a convex pentagon, but whether or not it is possible to select n points from any given $2^{n-2}+1$ points in the plane in such a way that the n points determine a convex n-gon is unknown.

II. *Solution by D. K. Pease, University of Connecticut.* We will consider the generalized problem: Let there be given $(n+3)$ points in an n-flat. Prove that we can select $(n+2)$ of them which determine a convex polytope.

We exclude cases where any $(m+1)$ points lie in an m-flat.

Choose any $(n+1)$ of the points. These determine an n-simplex. Now choose either of the remaining points. There will be three cases: (A) The point is within the simplex. (B) The point is within a vertical angle of the simplex. (C) The point determines, with the $(n+1)$ points of the simplex, a convex polytope.

For (C) there is nothing further to show.

(B) is in the same situation as (A) since the point at the vertex angle is within a simplex determined by the $(n+2)$nd point and the other n points.

We now have a simplex with a point inside. The point inside, together with any n points of the simplex, determines $(n+1)$ sub-simplexes.

Now consider the $(n+3)$rd point. It will also fall into cases (A), (B) and (C) with respect to the simplex. For case (A) the point will be within one of the sub-simplexes. This sub-simplex is made up of the vertical angles of the n other sub-simplexes. For case (B) the vertical angle of the simplex is made up of the vertical angles of n of the sub-simplexes. So, for (A) or (B) the $(n+3)$rd point will be in case (C) for $(n-1)$ of the sub-simplexes.

Also solved by Paul Brock, William Gustin, Frank Hawthorne, J. B. Kelly, Norman Miller, and C. R. Perisho.

A Pythagorean Inequality

E 741 [1946, 532]. *Proposed by William Scott, Columbus, Ohio*

Prove that in a (non-degenerate) right spherical triangle with hypotenuse c and legs a, b we have $a^2+b^2>c^2$.

I. *Solution by the Proposer.* We may assume $0<a\leqq b<\pi$, $0<c<\pi$. Also, we have $\cos c=\cos a \cos b$.

If $c\geqq\pi/2$, then $\cos c\leqq 0$. Therefore, for this case, $\cos b\leqq 0$, $\cos c\geqq\cos b$, $c\leqq b$, and $a^2+b^2>c^2$.

Thus we may take $c<\pi/2<\sqrt{8}$. Let b be fixed and note that

$$dc/da = (\sin a \cos b)/\sin c > 0.$$

Let $b<c\leqq\sqrt{2}b$. Then, since $0<c<\sqrt{8}$,

$$(\sin c)/c > 1 - c^2/6 > 1 - c^2/4 + c^4/96 > \cos (c/\sqrt{2}) \geqq \cos b.$$

Therefore

$$d(a^2 + b^2)/da = 2a > 2c \cos b \sin a/\sin c = d(c^2)/da.$$

Since $c=b$ when $a=0$, this shows that $a^2+b^2>c^2$ for $b<c\leqq\sqrt{2}b$. Hence if

$c=\sqrt{2}b$, then $a>b$. Therefore, since $da/dc>0$, if $c>\sqrt{2}b$, then again $a>b$.

Hence $a^2+b^2>c^2$ for any b and all a such that $0<a\leqq b$, and the proof is complete.

II. *Solution by L. M. Kelly, University of Missouri.* The announced relation follows quite nicely from a theorem of L. M. Blumenthal: *If a spherical triangle is reproduced congruently in the plane (that is, with length of sides preserved), each angle of the plane triangle is less than the corresponding angle of the spherical triangle.* In the particular case at hand, reproduction of the right spherical triangle in the plane yields a plane triangle in which angle C is acute, from which it follows, by the cosine law, that $c^2<a^2+b^2$.

It might be well to indicate a simple proof of Blumenthal's theorem which, it would seem, should be better known than is the case.

Let ABC be the spherical triangle and $A'B'C'$ the corresponding plane triangle, where $BC=B'C'=a$, $CA=C'A'=b$, $AB=A'B'=c$. Now

$$\sin\frac{A'}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\sin\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}.$$

Since $A/2$ and $A'/2$ are certainly less than or equal to $\pi/2$, the theorem will be proved if we can show that the right member of the first equation is less than that of the second. This we can do in the following way. First note that $(\sin x)/x$ is a montone decreasing function for $0<x<\pi/2$. Furthermore, since $a<b+c$, $s-b<c$. Similarly $s-c<b$. Thus

$$\frac{\sin(s-b)}{s-b}\cdot\frac{\sin(s-c)}{s-c}>\frac{\sin b}{b}\cdot\frac{\sin c}{c},$$

or

$$\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}>\frac{(s-b)(s-c)}{bc}.$$

In a precisely analogous fashion we may prove that reproduction in the plane of a pseudo-spherical triangle (a geodesic triangle on the pseudo sphere) results in correspondingly larger angles. These results may be phrased, of course, in terms of congruent imbedding in spherical and hyperbolic spaces. Furthermore, these two results have been extended by Blumenthal as follows: *If a "shortest distance" triangle on a surface having everywhere positive curvature be reproduced congruently in the plane, the angles of the plane triangle will each be less than the corresponding angle of the surface triangle. A similar result holds for surfaces of everywhere negative curvature.* It is necessary to make a careful distinction between geodesic and shortest distance triangles, since they are not always the same.

Some Tangent Circles of a Triangle

E 742 [1946, 532]. *Proposed by B. F. Laposky, Cherokee, Iowa*

Let ABC be a triangle, LMN the median triangle, DEF the orthic triangle, O the circumcenter, J the nine-point center, and T, U, V the other intersections of the medians AL, BM, CN with the nine-point circle (J). Now there are two sets of circles tangent to the circumcircle at the vertices A, B, C and also tangent to (J). Show that the circles of one set have their centers at the intersections of OA, OB, OC with the corresponding sides of LMN and touch (J) at D, E, F; the circles of the other set have their centers at the intersections of OA, OB, OC with the lines JT, JU, JV and touch (J) at T, U, V.

Solution by B. R. Leeds, Brooklyn, N. Y. Let S_1 be the intersection of DJ with OA, and P the other intersection of (J) with AD. Then, if H is the orthocenter, $HJ=JO$, and PJ is parallel to OA. Therefore $\sphericalangle OAD=\sphericalangle JPD=\sphericalangle JDP$, whence $S_1A=S_1D$. Now MN is the perpendicular bisector of the altitude AD, and therefore passes through S_1. Thus S_1 is the center of a circle (S_1) passing through A and D. Since S_1, J, D are collinear, (S_1) is tangent to (J) at D. Similarly, since S_1, O, A are collinear, (S_1) is tangent to (O) at A. In a similar manner circles (S_2), (S_3) can be constructed tangent to (J), (O) at E, B and F, C respectively.

Let C_1 be the intersection of JT with OA. Since $\sphericalangle ADB$ is a right angle, PJ, which is parallel to OA, passes through L, and $\sphericalangle C_1AT=\sphericalangle TLP=\sphericalangle LTJ=\sphericalangle C_1TA$. Thus C_1 is the center of a circle (C_1) passing through A and T. Since C_1, T, J are collinear, (C_1) is tangent to (J) at T. Similarly, since C_1, O, A are collinear, (C_1) is tangent to (O) at A. In a similar manner circles (C_2), (C_3) can be constructed tangent to (J), (O) at U, B and V, C respectively.

Also solved by E. R. Stabler and the proposer.

Number of Real Solutions of a Transcendental Equation

E 743 [1946, 532]. *Proposed by E. P. Starke, Rutgers University*

Determine the conditions on the constants a and b such that

$$\log x - ax + b = 0$$

shall have two real solutions, one real solution, or no real solutions.

I. *Solution by Norman Miller, Queen's University.* The question is that of the the points of intersection of the line $y=ax-b$ with the curve $y=\log x$. Inspection of the curve shows that any line with negative or zero slope cuts it in one point only. If $a>0$, three cases arise. The tangent with y-intercept $-b$ is found to be $y=xe^{b-1}-b$. It follows that, if $a=e^{b-1}$, the given equation has one real root, which may be regarded as a double root; if $a>e^{b-1}$, the equation has no real root; if $0<a<e^{b-1}$, the equation has two real roots; if $a\leqq 0$, the equation has one real root.

II. *Solution by C. F. Pinzka, North Plainfield, N. J.* Let

$$y = \log x - ax + b,$$

then $y' = 1/x - a$ and $y'' = -1/x^2$, the latter being always positive. If $a \leqq 0$, y' is always positive and, since y increases from $-\infty$ to $+\infty$, there is one real solution. If $a > 0$, $y' = 0$ for $x = 1/a$, and a maximum exists at $y = b - 1 - \log a$. Hence there are no, one, or two real solutions if $a > 0$ according as $b - 1 - \log a$ is less than, equal to, or greater than zero.

Also solved by D. W. Alling, P. T. Bateman, R. G. Blake, Paul Brock, F. L. Celauro, N. J. Fine, I. M. Gardoff, R. T. Hood, Meyer Karlin, L. M. Kelly, R. J. Koch, Sidney Kravitz, LeRoy Pietsch, C. F. Pinzka (also like I), A. Sisk, C. W. Topp, and the proposer.

A Set of n Positive Integers

E 744 [1946, 532]. *Proposed by Paul Erdös, Syracuse University*

Let $a_1 < a_2 < \cdots < a_n \leqq 2n$ be n positive integers such that the least common multiple of any two is greater than $2n$. Then $a_1 > [2n/3]$.

I. *Solution by N. J. Fine, Washington, D. C.* It is clear that no one of the numbers can divide another. Hence, writing $a_i = 2^{p_i} A_i$, A_i odd, we see that the A_i are all different. Since there are n of them, they coincide in some order with the set of all odd numbers less than $2n$.

Now consider $a_1 = 2^{p_1} A_1$. If $a_1 \leqq [2n/3]$, then $3a_1 = 2^{p_1} \cdot 3A_1 \leqq 2n$ and $3A_1 < 2n$. Hence $3A_1 = A_j$ for some j, and $a_j = 2^{p_j} \cdot 3A_1$. The least common multiple of a_1 and a_j is either $2^{p_1} \cdot 3A_1 = 3a_1 \leqq 2n$ or $2^{p_j} \cdot 3A_1 = a_j \leqq 2n$. This contradiction proves the theorem.

II. *Solution by the Proposer.* Suppose $a_1 \leqq [2n/3]$, then $3a_1 \leqq 2n$. Consider $2a_1, 3a_1, a_2, \cdots, a_n$, a set of $n+1$ integers no one of which divides another. This is impossible, whence the theorem.

Also solved by Murray Barbour, P. T. Bateman, N. Kaufman, and J. M. Zucker.

A Theorem on Arithmetic Progressions

E 745 [1946, 532]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If, in an arithmetic progression of n terms and common difference d, n is prime to each term, then n is not prime to d.

Solution by P. T. Bateman, Yale University. Suppose n were prime to d. Then, if the first term of the arithmetic progression is a, the numbers $a, a+d, a+2d, \cdots, a+(n-1)d$ would have different residues modulo n, and thus one of them would be divisible by n, which contradicts the hypothesis.

Also solved by Murray Barbour, Paul Brook, N. J. Fine, William Gustin, R. T. Hood, V. L. Klee, Jr., C. F. Pinzka, Robert Seall, E. D. Schell, David Wellinger, and the proposer. A number of solvers pointed out that this is merely a rewording of a theorem known to Euler. See Dickson, *History of the Theory of Numbers*, p. 113, and *Introduction to the Theory of Numbers*, Theorem 9, p. 6.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins of at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4249. *Proposed by W. B. Campbell, Philadelphia Textile Institute*

A body is projected from a point O in a plane making an angle A with the horizontal, the direction of projection being in a vertical plane containing a line of greatest slope of the plane, and making an angle B with the upward direction of that line. If the plane be smooth and the body perfectly elastic, derive expressions for t_n, the time consumed in the nth flight, and for x_n, the coördinate of the point of impact at the end of the nth flight. Will it ever strike O again, and will any of its flights be vertical? What is maximum x_n?

4250. *Proposed by Richard Bellman, Princeton University*

If

$$s_n = \sum_{k=1}^{n} a_k, \qquad \sigma_n = \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) a_k,$$

and

$$\sum_{n=1}^{\infty} |s_n - \sigma_n|^k < \infty,$$

for any $k>0$; prove that $\sum_{n=1}^{\infty} a_n$ is convergent.

4251. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In an orthocentric tetrahedron the lines joining the symmedian points (Lemoine points) of the faces to the midpoints of the corresponding altitudes, are concurrent at a point such that the sum of the squares of its distances to the planes of the faces is a minimum.

4252. *Proposed by Paul Erdös, Syracuse University*

It is well known that $2n!/n!(n+1)!$ is always an integer. Prove that for every k there are infinitely many n's such that $2n!/n!(n+k)!$ is an integer.

4253. *Proposed by G. T. Williams, Cambridge, Massachusetts*

Given two tangent unit circles, C_1 and C_2, and their common external tangent, T. A third circle, C_3, is drawn tangent to C_1, C_2, and T; C_4 is then drawn

tangent to C_1, C_2 and C_3; and so on, each C_{i+1} being tangent to C_1, C_2, and C_i. Find the total area of the aggregate of circles, C_1, C_2, C_3, · · · ·

SOLUTIONS

Polars and a Family of Circles

3962 [1940, 323]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A circle (C) with a given radius rolls on a fixed circle (C'). Find the form of the locus of the points of intersection of (C) with the polar of a fixed point with respect to (C). Consider the cases where (C') reduces to a point, or a straight line.

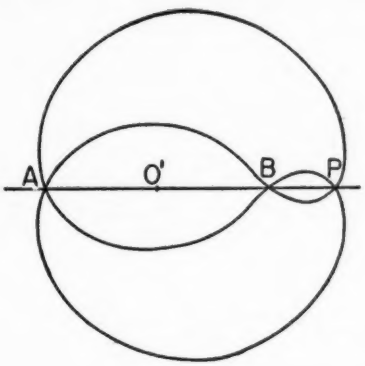

FIG. 1

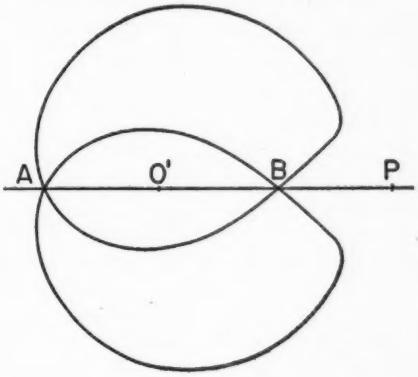

FIG. 2

Solution by R. Bouvaist, Vincelles, Saône-et-Loire, France. The origin is the center O' of the fixed circle (C'). The axis $O'X$ is directed through P, with $O'P=a$. The locus of the center of (C) is a circle with center O': let R be its radius, and let r be the radius of (C). The family of circles (C) is then given by

$$(x - R\cos\phi)^2 + (y - R\sin\phi)^2 = r^2.$$

The points of contact of the tangents to (C) through P are on the circle of diameter PO (O being the center of (C)) whose equation is

$$(x - a)(x - R\cos\phi) + y(y - R\sin\phi) = 0,$$

whence the desired locus has the parametric representation

$$\frac{x - R\cos\phi}{y} = \frac{y - R\sin\phi}{a - x} = \frac{\pm r}{[(x-a)^2 + y^2]^{1/2}}. \tag{1}$$

Translating the origin to P, and using polar coördinates, we obtain the equation of the locus in the form

$$(\rho + a\cos\theta)^2 = (R - r \pm a\sin\theta)(R + r \mp a\sin\theta).$$

This equation defines a tricircular sextic having three double-points at a finite distance on $O'X$: P and the points A and B symmetric with respect to O' and such that $O'A = O'B = (R^2 - r^2)^{1/2}$. Furthermore the curve has $O'X$ as axis of symmetry. It is not difficult to sketch. Figure 1 shows such a curve for the case $r < R < a < R + r$. P becomes an isolated double-point when either $a < R - r$ or $a > R + r$; the latter case is shown in Figure 2. If the radius of (C') is zero, so that $R = r$, the two double-points A and B are brought into coincidence at O' so that the two branches of the curve are tangent to $O'X$ at O'. If $R < r$, all three double-points are isolated and the locus degenerates into two separate ovals. The resulting figures are easy to visualize from those shown.

When the circle (C') is replaced by a line (L), analysis similar to the above yields the equation

$$(x^2 + y^2)(x - r + a)^2 = r^2 y^2$$

or

$$\rho \cos \theta = r(1 \pm \sin \theta) - a,$$

where the origin is taken at P and (L) is the line $x + a = 0$. This circular quartic has double-points at P and at $A: (r - a, 0)$, and has the lines $x + a = 0$ and $x + a = 2r$ as asymptotes. Figure 3 shows the locus for the case $r < a < 2r$. If $a > 2r$ or $a < 0$, then P becomes an isolated double-point.

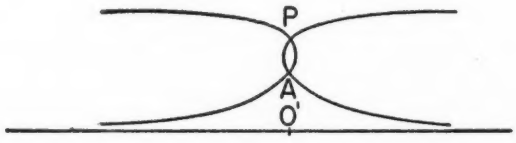

FIG. 3

The Lost Hunter

4185 [1946, 44]. *Proposed by B. M. Stewart, Michigan State College*

In an unexplored region known as Wild Basin, a hunter found himself lost. But he had on hand a compass, and there were visible on two distant peaks fire ranger stations, A and B, whose bearings from his own cabin, O, he knew. From one observation point, C, the hunter took bearings on A and B; walking to another observation point, D, nearby, he took bearings on A, B, and C. Somehow he felt these seven bearings ought to enable him to find the direction homeward, that is, the bearing from D to O.

Show that the hunter's problem may be solved if he has either (1) mathematical tables or (2) a straightedge, in this case using the compass card as a protractor.

Note. Suggested by a correspondent of M. H. Ingraham, University of Wisconsin.

Solution by E. S. Keeping, University of Alberta, Edmonton. Let θ be the bearing of O from D; α, β, the bearings of A, B from O; α_1, β_1, the bearings of A, B from C; α_2, β_2, γ_2, those of A, B, C from D. We seek to determine θ.

(1) All the angles of the triangles ODA, DAC, DCB, ODB are determined in terms of the seven known bearings and θ. The sine law and the identity

$$\frac{OD}{DB} = \frac{OD}{DA}\cdot\frac{DA}{DC}\cdot\frac{DC}{DB}$$

give

$$\frac{\sin(\beta-\beta_2)}{\sin(\theta-\beta)} = \frac{\sin(\alpha-\alpha_2)}{\sin(\theta-\alpha)}\cdot\frac{\sin(\alpha_1-\gamma_2)}{\sin(\alpha_1-\alpha_2)}\cdot\frac{\sin(\beta_1-\beta_2)}{\sin(\beta_1-\gamma_2)},$$

whence

$$\sin(\theta-\alpha) = k\sin(\theta-\beta),$$

$$k = \frac{\sin(\beta-\beta_2)\sin(\alpha_1-\alpha_2)\sin(\beta_1-\gamma_2)}{\sin(\alpha-\alpha_2)\sin(\beta_1-\beta_2)\sin(\alpha_1-\gamma_2)}.$$

From these results the required bearing θ may be obtained. We find

$$\tan\theta = \frac{\sin\alpha - k\sin\beta}{\cos\alpha - k\cos\beta}.$$

(2) Without tables, but with a protractor and straight edge, the problem may be solved by the following easy construction. Lay off CD of any convenient length. The lines CA, DA, CB, DB locate points A and B. Then the lines AO and BO determine O, whereupon the bearing of DO can be read.

In two cases, there is no solution by construction, and the trigonometric equation becomes indeterminate: (1) if CD is collinear with A (or B), or (2) if O is collinear with AB. If no further facts are available, the hunter may still take the long way home: follow the known bearing to A (or B) and then from there home to O.

Solved also by M. H. Ingraham, Vladimir Karapetoff, W. A. Rees, R. H. Urbano, and the Proposer.

Monge Point and Circumcenter of a Tetrahedron

4187 [1946, 45]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

For a given tetrahedron the ratio of the distances of the Monge point and of the circumcenter to the common perpendicular to two opposite edges is equal to $\cos\theta$, where θ is the angle between the two edges.

Solution by P. D. Thomas, Navy Department, Washington, D. C. If $ABCD$ is the given tetrahedron, let the feet of the common perpendicular to the sides AB, CD be P, Q, respectively, and take $PQ=2a$, $AP=d$, $PB=b$, $DQ=e$, $QC=c$. At S, the midpoint of PQ, construct a plane t perpendicular to PQ. Let the inter-

nal and external bisectors of the angle θ between the projections of AB, CD upon t be the x- and y-axes, respectively. PQ is then the z-axis. The coördinates of the vertices are then

$$A(-d\cos R, -d\sin R, a), \qquad B(b\cos R, b\sin R, a),$$
$$C(c\cos R, -c\sin R, -a), \qquad D(-e\cos R, e\sin R, -a),$$

where $R=\frac{1}{2}\theta$.

Let the Monge point and the circumcenter be respectively $M(x_1, y_1, z_1)$, $O(x_2, y_2, z_2)$. Then d_1, d_2, the distances to PQ (the z-axis) from M, O, respectively, are given by

$$d_1^2 = x_1^2 + y_1^2, \qquad d_2^2 = x_2^2 + y_2^2.$$

The coördinates of O and M are easily determined as

$$O\left(\frac{b+c-e-d}{4}\sec R, \frac{b+e-c-d}{4}\csc R, \frac{bd-ce}{4a}\right),$$

$$M\left(\frac{b+c-e-d}{4}\sec R\cos 2R, \frac{c+d-b-e}{4}\csc R\cos 2R, \frac{ce-bd}{4a}\right).$$

Hence $x_1=x_2\cos 2R$, $y_1=y_2\cos 2R$, and the desired result follows at once.

Solved also by M. R. Blanchard, R. Bouvaist, and the Proposer.

Editorial Note. The proposer remarks that the distance from the circumcenter to the common perpendicular of the two edges AB and CD, multiplied by $\sin\theta$, is equal to the projection on a plane parallel to AB and CD of the line joining the midpoints of AB and CD. A similar statement holds in which we replace the circumcenter by the Monge point and the factor $\sin\theta$ by $\tan\theta$. In case the tetrahedron is orthocentric, so that $\cos\theta=0$, we come back to a known property. (See N. A. Court, *Modern Pure Geometry*, p. 63.)

Orthogonal Sets of Integers

4190 [1946, 103]. *Proposed by Norman Anning, University of Michigan.*

If a, b, c, R are integers such that $a^2+b^2+c^2=R^2$, solve in integers the simultaneous equations

$$x^2 + y^2 + z^2 = R^2$$
$$ax + by + cz = 0.$$

There are at least four solutions

Solution by Fritz Herzog, Michigan State College Since the stated equations are homogeneous it is evidently sufficient to consider the case in which a, b, c have no common divisor, that is,

$$(a, b, c) = 1. \tag{1}$$

We may as well assume that $abc\neq 0$, for if one, say c, vanishes, we have at once the desired solutions, namely, $x=\pm b$, $y=\mp a$, $z=0$ and $x=0$, $y=0$, $z=\pm R$.

Let x, y, z be a solution of the problem. We have then

$$z = -(ax + by)/c, \tag{2}$$

$$c^2(x^2 + y^2) + (ax + by)^2 = c^2R^2. \tag{3}$$

Multiplying (3) by b^2+c^2, we obtain, after simple algebraic changes,

$$c^2R^2x^2 + [abx + (b^2 + c^2)y]^2 = (b^2 + c^2)c^2R^2.$$

We put

$$X = x, \qquad Y = \frac{abx + (b^2 + c^2)y}{cR}, \tag{4}$$

so that

$$X^2 + Y^2 = b^2 + c^2. \tag{5}$$

It is easily seen that, on the other hand, any solution in integers X, Y of (5) leads to values x, y, z, determined by (4) and (2), which will solve the problem provided only that the value of y from (4) is integral, that is,

$$abX \equiv cRY \pmod{b^2 + c^2}. \tag{6}$$

By (2) and (3), z will then be an integer. It is also clear from (4) that two different solutions X, Y of (5) and (6) will lead to two different pairs of values x, y and thus to two different solutions x, y, z of the problem. It remains therefore to show that there are at least four pairs of integers X, Y which satisfy (5) and (6).

To show this, we begin with the congruence

$$a^2b^2 \equiv -c^2R^2 \pmod{b^2 + c^2}, \tag{7}$$

which follows at once from $a^2 \equiv R^2$ and $b^2 \equiv -c^2 \pmod{b^2+c^2}$. Now the greatest common divisor of a^2b^2 and b^2+c^2 must be a square. For if this were not the case there would exist a prime p and a positive integer j such that p^{2j-1} is the highest power of p which divides b^2+c^2, while p^{2j} divides a^2b^2. Since by (1), p could not be a prime divisor of both a and b, we would have p^{2j} divides either a^2 or b^2. However, the latter would imply that p^{2j-1} would be the highest power of p that divides c^2, while the former (together with $b^2+c^2 = R^2-a^2$) would imply that p^{2j-1} is the highest power of p that divides R^2. Both of these alternatives are impossible.

We may therefore write $(a^2b^2, b^2+c^2) = d^2$ and obtain from (7)

$$(ab/d)^2 \equiv -(cR/d)^2 \left(\mathrm{mod}\,\frac{b^2 + c^2}{d^2}\right). \tag{8}$$

Since (ab/d) is relatively prime to the modulus, we can determine λ such that

$$\lambda(ab/d) \equiv (cR/d) \left(\mathrm{mod}\,\frac{b^2 + c^2}{d^2}\right). \tag{9}$$

From (8) and (9) we conclude that

$$\lambda^2 \equiv -1 \left(\mod \frac{b^2+c^2}{d^2}\right).$$

Therefore, according to a well-known theorem,* it is possible to find integers u and v such that

$$u^2 + v^2 = \frac{b^2+c^2}{d^2}, \tag{10}$$

$$u \equiv \lambda v \left(\mod \frac{b^2+c^2}{d^2}\right). \tag{11}$$

From (9) and (11) we have

$$(ab/d)u \equiv (cR/d)v \left(\mod \frac{b^2+c^2}{d^2}\right). \tag{12}$$

Multiplying (10) and (12) throughout by d^2 and putting $ud = X$ and $vd = Y$, we obtain the fact that X, Y is a solution of (5) and (6). Finally, since the four pairs (u, v), $(-u, -v)$, $(v, -u)$ and $(-v, u)$ are easily seen to be four different solutions of (10) and (11), we obtain from them the four solutions (X, Y), $(-X, -Y)$, $(Y, -X)$ and $(-Y, X)$, respectively, of (5) and (6).

Solved also, partially, by Murray Barbour, Mary A. English, and the Proposer.

Editorial Note. From equations (2) and (4) of the above solution we easily derive

$$Y = \frac{cy - bz}{R}.$$

Put $Y = -x'$, and by analogous argument obtain the integers

$$y' = \frac{cx - az}{R}, \qquad z' = \frac{ay - bx}{R}.$$

It is now easy to show that the determinant

$$\begin{vmatrix} a & b & c \\ x & y & z \\ x' & y' & z' \end{vmatrix} \tag{13}$$

has the following properties: the sum of the squares of the elements in any row or column is R^2, any two rows or two columns are orthogonal, the cofactor of each element a_{ij} equals Ra_{ij}, the value of the determinant is R^3. It follows that $\pm x$, $\pm y$, $\pm z$ and $\pm x'$, $\pm y'$, $\pm z'$ provide four solutions of the original problem.

* See, for instance, Landau, Vorlesungen über Zahlentheorie, vol. 1, p. 101, Theorem 160.

The general solution in integers of the equation

$$a^2 + b^2 + c^2 = R^2$$

is given by

$$a = \tfrac{1}{2}(p^2 + q^2 - m^2 - n^2), \qquad b = np - mq,$$
$$c = mp + nq, \qquad R = \tfrac{1}{2}(p^2 + q^2 + m^2 + n^2).$$

(See Carmichael, *Diophantine Analysis*, pp. 35–44.) In terms of m, n, p, and q, it is now easy to find explicit expressions for the elements of (13). They are

$$x = np + mq, \qquad y = \tfrac{1}{2}(n^2 + q^2 - m^2 - p^2), \qquad z = mn - pq$$
$$x' = mp - nq, \qquad y' = mn + pq, \qquad z' = \tfrac{1}{2}(m^2 + q^2 - n^2 - p^2).$$

The Proposer noted that essentially this same determinant is mentioned in Coxeter, *Non-Euclidean Geometry*, p. 126; and in Kovalevski, *Determinanten*, p. 177.

The interesting possibility of extension to higher orders invites further investigation. The determination of x_i, $i=1, 2, \cdots, n$, for given a_i where $\sum a_i^2 = R^2$, such that $\sum a_i x_i = 0$ and $\sum x_i^2 = R^2$ is always possible when n is even, in at least a trivial way. Among the examples that have come to hand, the smallest value of n for which x_i do not exist for certain a_i, is $n=9$: if the a_i are all odd (*e.g.* 3, 3, 1, 1, 1, 1, 1, 1, 1) there must be an odd number of odd x_i, whence $\sum a_i x_i$ cannot be zero.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Analytic Geometry and Calculus. By J. F. Randolph and Mark Kac. New York, The Macmillan Co., 1946. 642 pages (including tables and answers). $4.75.

This text for a unified first course is designed to present the basic notions of the calculus as early as possible. After a brief treatment of the straight line and circle it proceeds to functions, derivatives and their applications, differentials, definite integrals, and the fundamental theorem, dealing always with explicit algebraic functions. This development occupies the first third of the book. Only after this does it take up conics, change of coördinates, and even such topics as perpendicularity and the angle between two lines. Parametric equations and polar coördinates come still later.

This arrangement of material has two advantages. It makes the elements

of the calculus available for use in science and engineering courses much earlier than the conventional order, and it permits a more satisfactory treatment of some topics in analytic geometry, notably asymptotes and general properties of graphs. Most of the usual material in analytic geometry is covered, but the treatment is very condensed; for example, tangents and focal properties of conics are left as exercises.

The text is printed in two sizes of type. The sections in large print constitute a fairly formal presentation. More rigorous proofs and a good deal of material commonly left to more advanced courses are included as sections in small print intended for the student who wishes to go deeper. The authors suggest that 10 semester hours should suffice to cover the sections in large print. Chapters on Taylor's theorem, solid analytic geometry, partial derivatives, and multiple integrals are included, but not on infinite series or differential equations.

The exposition is for the most part very clear, and a sincere effort is made to distinguish between what is proved and what is assumed. Problems are grouped according to difficulty and tend to avoid technical complication.

A number of more or less novel features may be noted. The symbols Δx and dx are defined as arbitrary numbers, not necessarily equal, which are distinguished by their specific uses, Δx as a number to be added to a value of the independent variable, dx as a number to be multiplied by $f'(x)$. Anti-derivatives are introduced very early, but the integral sign is not used until the definite integral is defined. Area and arc length are defined from the outset by limiting processes. The emphasis at the very beginning on algebraic manipulation of inequalities and absolute values is an excellent feature, as is the varied use of the "greatest integer" function $[x]$. The notation $\ln x$ for $\log x$ is adopted.

One feature of the arrangement that may be criticized is the postponement of the formula for the derivative of a composite function until very late (page 207). As a result, the method of integration by substitution cannot be properly justified when it is first used, the rule for differentiating implicit functions is given without its rational explanation, and the formula for the derivative of u^p, for fractional p, has to be assumed without proof for many chapters. Another questionable feature is the presentation of the notion of definite integral in simplified form, with equal subdivisions and the value of the function taken at an end-point of each. The more general formulation is mentioned in small print but is seldom used. With the less flexible definition one is not free to take the approximating sums equal to the quantity being computed, and this tends to obscure somewhat the precision of integral calculations. The general definition is just as easy to teach and is superior both in theory and application. Of course, these features can be corrected by a teacher if he wishes.

As a whole, this is definitely a superior text that should prove especially useful where early introduction to the calculus is desired. It should also be noted that the text can be adapted to a course in calculus for students who have already had a course in analytic geometry, as the authors point out in the preface.

J. C. OXTOBY

An Introduction to College Mathematics. By C. V. Newsom. New York, Prentice-Hall, Inc., 1946. 8+344 pages. $3.50.

The problem of how and what to teach the non-specialist student who takes one year of college mathematics and no more has long rested uneasily upon the conscience of most teachers of mathematics. What information, what ideas should such a student carry away from his last contact with mathematics? Most of us would agree wholeheartedly with the objectives set forth forty years ago in the Meran curriculum:

"1. A scientific survey of the systematic structure of mathematics.

2. A certain degree of skill in the complete handling, numerical and graphical, of problems.

3. An appreciation of the significance of mathematical thought for a knowledge of nature and for modern culture."

To these we might add Godfrey's statement that "we want to make boys believe that mathematics is indispensable in their daily life and not something which they will have to do in hell."

That the traditional course of algebra, trigonometry, and analytic geometry, either seriatim or in any "unified" permutation, leaves something to be desired in aiding the attainment of these objectives can hardly be questioned. Several text-books of the "cultural" type have been written in a laudable attempt to solve the problem of imparting at least the rudiments of a liberal mathematical education in one year. At present, however, even the existence of a solution of this problem has yet to be convincingly demonstrated. Nevertheless, each new text that is written for the non-specialist student does make its peculiar contribution toward the solution of the problem, and the text under review is a welcome and valuable addition to the lists.

Professor Newsom states that his book "has been written for the student and not for the instructor." Knowing that the student's view of the course will depend in large measure upon the problems he works, the author has expended unusual effort and care in devising over 800 problems. The field of application in the many practical problems ranges from cometary orbits to chimney area, from the velocity of ants to that of the circulation of money. Many of the exercises "are important in the development of the argument," and the student is warned not to omit these.

The choice of material is a matter on which there has been, perhaps fortunately, little unanimity among authors of "cultural" texts. Professor Newsom explains that "No topic has been chosen simply because of some special appeal to the author or because of a traditional prejudice among mathematicians in favor of it. In fact, all material introduced was examined critically for its possible value to the non-specializing student. . . . Considerable material proposed for the course was discarded when it was found that students generally were unable to appreciate its significance." About 12 per cent of this rather slim book (301 pages plus tables, answers, index) is devoted to topics such as logic, bases of numeration, infinite sets, that are not ordinarily encountered in the traditional first-year

text. The choice and order of the subject matter are indicated by the chapter headings: 1. The Nature of Mathematics, 2. Number and the Operations of Arithmetic, 3. The Arithmetic of Numbers in the Exponential Form, 4. The Arithmetic of Measurement, 5. Logarithms, 6. Some Topics in the Mathematics of Finance, 7. Progressions of Numbers, 8. Combinations and Probability, 9. Functional Relationships, 10. Variation, 11. The Circular Functions, 12. The Equation, 13. Some Common Curves. Some will quarrel with the author's decision to discuss compound interest and annuities before taking up geometric progressions.

An *Introduction to College Mathematics* has none of the florid rhetoric with which some texts are made obscure. Both chapter and section headings are clear and helpful signposts and not coy conundrums set by the author. Professor Newsom writes lucidly in a pleasingly informal, conversational style; his book will be welcomed by all teachers of non-specialist students.

R. D. SPECHT

NEW BOOKS RECEIVED

Cosmic Radiation. Fifteen Lectures. Edited by W. Heisenberg. Translated from the German by T. H. Johnson. New York, Dover Publications, 1946. 192 pages. $3.50.

Cours Complet de Mathématiques Élémentaires. Tome 1. *Arithmetique.* By J. Haag. Paris, Gauthier-Villars, 1945. 6+105 pages. 80 Fr.

Éléments de Calcul Infinitésimal. By Adrien Grosrey. Paris, Gauthier-Villars, 1945. 192 pages. 280 Fr.

Éléments de Mathématiques Supérieures. By P. Gaudiot. Paris, Editions Leon Eyrolles, Librairie de l'Enseignement Technique, 1944. 381 pages. 182 Fr.

Essential Business Mathematics. By L. R. Snyder. New York and London, McGraw-Hill Book Co., Inc., 1947. 12+434 pages. $2.75.

Introduction to Mathematical Statistics. By P. G. Hoel. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1947. 10+258 pages. $3.50.

Matrix and Tensor Calculus with Applications to Mechanics, Elasticity, and Aeronautics. By A. D. Michal. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1947. 13+132 pages. $3.00.

Meson Theory of Nuclear Forces. By W. Pauli. New York, Interscience Publishers, Inc., 1946. 7+69 pages. $2.00.

Vorlesungen über Differential- und Integralrechnung. Erster Band. Funktionen einer Variablen. By A. Ostrowski. Basel, Birkhauser, 1945. 12+373 pages.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to L. F. Ollmann, Hofstra College, Hempstead, New York.

EDITOR'S NOTE.—A letter asking for a report of club activities for 1946–47 has been sent by the editor of this department to the sponsors of the various mathematical clubs and societies. A general invitation is hereby extended to all the readers of this section to submit articles, suggestions, and papers to the editor which might be of interest to the members of mathematical clubs. Information on sources of mathematical films, slides, and models as well as bibliographies on topics suitable for discussion in meetings are particularly desired. This material will be collected for dissemination to individuals upon request and will be summarized in this department of the MONTHLY at intervals.

CLUB REPORTS, 1945-46

Pi Mu Epsilon, University of Illinois

The theme for the activities of the *Illinois Alpha* chapter of *Pi Mu Epsilon* was *The use of mathematics in other departments*. Staff members of various colleges of the University of Illinois addressed the forty-five graduate and undergraduate members on this theme as it applied to their respective fields.

The annual Spring banquet and initiation was held on May 16, at which time thirty-two students were initiated. The group was addressed by Professor Neiswanger of the Economics Department as he contributed to the general theme with the topic *The use of statistical methods in price control*. The annual *Pi Mu Epsilon* scholarship award was presented at this banquet to John Schumacher.

The officers for 1945–46 were: President, Clarence Phillips; Vice-President, Mildred Brannon; Secretary, LaVerne Bloomberg; Treasurer, Aileen Hostinsky; Faculty Advisor, Professor E. Welker.

Mathematics Club, Mount Mary College

The project for the year of the *Mathematics Club* of Mount Mary College was a survey of prospects in the business, scientific, educational and other fields for graduates with interest and ability in mathematics. Reports on results were given at each monthly meeting of the club. The average attendance at the meetings was twenty-seven. New members were initiated into the club after each had presented a summary of the life of a great mathematician.

The topics of the guest speakers appearing during the year include:

The atomic bomb, by Reverend Joseph F. Carroll, head of the Physics Department at Marquette University.

A search for perfection, by Dr. Harvey P. Pettit, head of the Mathematics Department of Marquette University. This topic dealt with a phase of the development of mathematics.

At another meeting a motion picture was shown which demonstrated Einstein's theory of relativity. The members of the club were hostesses for the an-

nual meeting of the *Wisconsin Section of The Mathematical Association* which was held at Mount Mary College on May 4. Sister Mary Felice of the Mount Mary College Mathematics Department was the presiding officer at this meeting.

Officers for 1945–46 were: President, Patricia O'Neill; Secretary-Treasurer, Marilyn Welsch; Club Adviser, Sister Mary Felice.

Mathematics Society, Brooklyn College

After a period of suspended activities, the *Mathematics Society* of Brooklyn College resumed meetings in February, 1946. The activities of the club centered around lectures on different phases of mathematics and the allied fields given by various members of the society. The topics were:

Summation of infinite series, by Dr. H. F. MacNeisch
Foundation of the number system, by Arthur Zeichner
Real sequences, by Julian Keilson
Transcendental numbers, by George Shapiro
Bruns method in the theory of numbers, by Gerard Washnitzer
Aerodynamics, by Dr. Moses Richardson
Various topics in mathematics, by Dr. R. A. Johnson
Vector and function spaces, by Gerard Washnitzer
Finite geometry, by Dr. James Singer
Prominence of Brooklyn College graduates in the field of Mathematics, by Dr. H. F. MacNeisch
Topology, by Daniel Waterman
Symbolic logic, by Melvin Hausner
Sense in curves, by Dr. Samuel Borofsky
Topics in mathematical logic, by Dr. Ira Rosenbaum of the Department of Philosophy.

A social evening in the form of a party was held with the Physics Society as one of the meetings. A semi-annual Integration Contest among the students of integral calculus was conducted by the club. The fourteenth edition of *The Math Mirror*, in which the high standard of the previous issues was maintained, was published in the Spring of 1946.

The officers for the February term of 1946 were: President, Gerard Washnitzer; Vice-President, Arthur Zeichner; Secretary, Cecile Pollack.

The officers elected for the September, 1946 term were: President, Julian Deilson; Vice-President, George Shapiro; Secretary, Rita Schwartz.

NEWS AND NOTICES

EDITED BY B. W. JONES, Cornell University

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

CANADIAN CONGRESS

The Canadian Mathematical Congress is planning a Summer Seminar to be held at the University of Toronto from August 15 to September 14, 1947. This year the Seminar will center on Algebra and Theory of Numbers and have as its object the stimulation of research in these fields on the part of younger mathematicians. Professor L. J. Mordell of Cambridge University, Professor Saunders Mac Lane of Harvard University, and Professor Paul Dubreil of the University of Nancy, will each give a series of eight lectures. The seminars in Algebra and Theory of Numbers will be presided over by Professor Richard Brauer of the University of Toronto and Professor Gordon Pall of the Illinois Institute of Technology, respectively.

The Seminar will be open to all qualified mathematicians on payment of a fee of $10.00. Arrangements will be made if possible for the accommodation of mathematicians and their families in one of the residences of the University at a moderate rate. Inquiries may be directed to the Secretariat, Canadian Mathematical Congress, Engineering Building, McGill University, Montreal, or to the secretary of the local committee, Professor G. de B. Robinson, Department of Mathematics, University of Toronto.

CONFERENCE ON ALGEBRA

The University of Michigan announces a Conference on Algebra to be held from Friday to Monday, July 25–28, 1947. Those who wish to present papers at the Conference, or who wish further information about it, are invited to communicate with R. M. Thrall, Department of Mathematics, University of Michigan. The program will feature papers by E. Artin, R. Brauer, S. Eilenberg, N. Jacobson and S. Mac Lane, and will have room for a considerable number of shorter papers. The time allotted to each paper will depend upon the total number of participants, but will in no case be less than 20 minutes. A limited number of dormitory rooms will be available to those attending the conference.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1947:

Case School of Applied Science. From June 30 to August 8 the following advanced courses will be offered: by Professor Agnew (Cornell University), infinite series and summability; by Professor Morris, introduction to higher geometry; by Professor Rinehart, introduction to algebraic theories.

Iowa State College. From June 16 to July 23 the following advanced courses

will be offered: by Professor Daniells, teaching of secondary school mathematics; by Professor Homeyer, design of experiments; by Dr. Mood, mathematical statistics; by Dr. Sealander, advanced calculus; by Professor Snedecor, statistical methods I. From July 24 to August 29: by Professor Kempthorne, advanced design of experiments, sampling methods; by Professor Snedecor, statistical methods II. From June 16 to August 29: by Dr. Anderson, advanced mathematics for engineers; by Dr. Gaskell, boundary value problems; by Dr. Rock, vector analysis; by Dr. Thielman, theory of functions of a real variable. During this twelve weeks' term a seminar in non-linear mechanics will be conducted and, if registration warrants, an additional graduate course will be offered.

Massachusetts Institute of Technology. From August 4 to September 19 there will be a special summer session program in applied mathematics in which the following graduate courses will be offered: by Professors P. D. Crout, F. B. Hildebrand, W. Prager, H. Reissner, J. L. Synge: advanced topics in applied mathematics; by Professors Hildebrand and E. Reissner, theory of plates and shells; by Professor C. C. Lin and E. Reissner, theoretical hydromechanics; by Professor D. J. Struik, tensors in mechanics.

Oklahoma Agricultural and Mechanical College. During the summer session the following series of lectures will be offered: by Professor Neugebauer of Brown University, history of mathematics; by Professor Lonseth of Northwestern University, theory of errors.

The University of Southern California. From June 23 to August 1 the following advanced courses will be offered: by Professor D. G. Bourgin (Illinois), Fourier series, partial differential equations; by Dr. Tobias Dantzig, history of mathematics; by Professor D. H. Hyers, differential equations; by Professor Ernst Snapper, theory of equations; by Professor D. V. Steed, tensor analysis; by Professor P. A. White, analytic geometry of space; by Professor R. L. Wilder (Michigan), mathematical analysis, seminar in topology.

The University of Wisconsin. From June 2 to September 20 the following graduate courses will be offered: advanced calculus, advanced college algebra, calculus of variations, college geometry, differential equations, higher algebra, higher analysis, modern theory of differential equations, vector analysis. From June 20 to August 15: advanced analytic geometry, differential geometry, higher analysis, introduction to the theory of probability, survey of the foundations of algebra. During the first period (summer semester) the following staff members will teach: Professors Allen, Bing, Everett, Gould, Kleene, Langer, MacDuffee, and Doctors Colvin, Sokolnikoff. During the second period (eight weeks' session) the following staff members will teach: Professors Arnold, Bing, Fullerton, Gould, MacDuffee, Whaples.

PERSONAL ITEMS

Professor Arnold Dresden of Swarthmore College was the delegate of the Mathematical Association at the meeting of the UNESCO at Philadelphia on March 24, 25, 26.

Professor Cristóbal de Losada y Puga of the Catholic University of Peru has been nominated by the President of the Republic of Peru as Minister of Public Education.

Professor Marston Morse of the Institute for Advanced Study was awarded the degree of Doctor (honoris causa) by the University of Paris on the recommendation of the Faculty of Science of the Sorbonne.

Assistant Professor Gordon Walker of Temple University will be the representative of the Mathematical Association at the meeting of the American Academy of Political and Social Science.

Dr. P. H. Anderson has been appointed economic analyst with the Marketing Division, Office of Domestic Commerce, Department of Commerce, Washington, D. C.

Assistant Professor Herbert Busemann of Smith College has been appointed to a professorship at the University of Southern California.

Assistant Professor Lola M. Christy of Westminster College, New Wilmington, Pennsylvania, has retired.

Assistant Professor B. H. Gere of the United States Naval Academy has been appointed to an associate professorship at Hamilton College.

Professor L. E. Gurney of the University of Southern California has retired.

Professor R. L. Krueger of Wittenberg College, Springfield, Ohio, has been appointed coordinator of the Division of Natural Sciences.

Associate Professor J. C. C. McKinsey of Oklahoma Agricultural and Mechanical College has been promoted to a professorship.

Assistant Professor E. N. Oberg of the University of Iowa has been promoted to an associate professorship.

The following appointments to instructorships are announced:

Queens College (tutorship): Jack Moshman

University of Arkansas: E. L. Eagle

Westminster College: W. A. Gibson

William Orange of Los Angeles City College died December 9, 1946.

Professor Emeritus W. A. Moody of Bowdoin College died February 3, 1947. He was a charter member of the Mathematical Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE IOWA SECTION

The thirty-third regular meeting of the Iowa Section of the Mathematical Association of America was held at Grinnell College, Grinnell, Iowa, on Friday and Saturday, April 19 and 20, 1946, in conjunction with the regular meeting of the Iowa Academy of Science. Professor E. N. Oberg, Vice-Chairman of the Section, presided.

There were thirty-five persons in attendance, including the following twenty-one members of the Association: E. W. Anderson, T. A. Bancroft, J. W. Beach, F. A. Brandner, E. W. Chittenden, W. M. Davis, Cornelius Gouwens, D. L. Holl, O. C. Kreider, F. M. McGaw, J. V. McKelvey, E. E. Moots, E. N. Oberg, H. V. Price, Fred Robertson, W. J. Rusk, E. R. Smith, L. W. Swanson, H. P. Thielman, Henry Van Engen, Roscoe Woods.

At the business meeting the following officers were elected for the coming year: Chairman, L. W. Swanson, Coe College; Vice-Chairman, H. V. Price, State University of Iowa High School; Secretary-Treasurer, Fred Robertson, Iowa State College.

The following papers were presented:

1. *College entrance requirements and their effect upon high school mathematics teaching*, by Professor Henry Van Engen, Iowa State Teachers College.

The speaker gave an illustration to show how the usual examination emphasizes operational procedure in solving problems. He urged that an understanding of the ideas involved should be stressed.

2. *Progress of demoted and promoted students in mathematics at Iowa State College*, by Professor Fred Robertson, Iowa State College.

Professor Robertson explained how the two groups (the demoted and the promoted) were selected, and presented tables showing the academic progress of these groups at his institution for the years 1936–1944 inclusive.

3. *Notes on some functional equations*, by Professor H. P. Thielman, Iowa State College.

It was shown that the equation

$$f(x)\cdot\phi(y) = X(x + y + nxy), \qquad n > 0,$$

can be reduced to $f(x)\cdot f(y) = f(0)\cdot f(x+y+nxy)$, and that all solutions of these equations, continuous for $x > -1/n$, are of the form $f(x) = a(1+nx)^b$, where a and b are constants. Two functions, $c(x)$ and $t(x)$, were defined as follows:

$$c(x) = \frac{1}{2}\left[f(x) + f\left(\frac{-x}{1+nx}\right)\right], \qquad t(x) = \frac{1}{2}\left[f(x) - f\left(\frac{-x}{1+nx}\right)\right]$$

where $f(x)=(1+nx)^b$. These functions were shown to possess many properties analogous to those of $\cos x$ and $\sin x$.

4. *Topological Groups*, by Professor Bernard Vinograde, Iowa State College, introduced by the Secretary.

This paper consisted of a description of the axiomatic foundations of topological group theory, and some immediate consequences and basic theorems.

5. *The converse of a certain theorem in analytic geometry*, by Professor Roscoe Woods, University of Iowa.

Professor Woods started with the well known fact that the three lines $a_ix+b_iy+c_i=0$, $i=1, 2, 3$, meet in a point if the determinant $|a_1b_2c_3|$ vanishes, and conversely (if the assumption is made that parallel lines meet in an ideal point). He then showed that by subjecting a given determinant $|A_1B_2C_3|$ to the so-called elementary transformations, there are transformations induced upon the coördinates x and y. These transformations represent the general collineation in the plane, so that any point associated with a given determinant $|a_1b_2c_3|=0$ may be transformed into any other point in the plane.

6. *The algebra of four functional operators on Boolean algebra and its characterization*, by Professor E. W. Chittenden, State University of Iowa.

This paper is to be published in the *Proceedings of the Iowa Academy of Science.*

7. *Approximate formulas for radii of circles including a proportion p of errors subject to a normal bivariate distribution*, by Professor E. N. Oberg, State University of Iowa.

This paper dealt with approximate formulas for the determination of the radius of a circle that includes a proportion p of errors from a normal bivariate distribution.

8. *Some new and old properties of the incomplete beta function*, by Professor T. A. Bancroft, Iowa State College.

Professor Bancroft gave several recurrence formulas relating to the incomplete "normalized" beta function. The possibility of their use in the extension of the *Incomplete Beta Function Tables* of Karl Pearson was discussed.

9. *Secondary mathematics in the post-war years*, by Professor H. V. Price, State University of Iowa High School.

It was stated that, in the opinion of the Commission on Postwar Plans of the National Council of Teachers of Mathematics, the first purpose of the high school mathematics program is to guarantee functional competence to all who can possibly achieve it. To attain this goal a combined program of conventional and general mathematics seems to be the answer.

10. *Some properties of differential equations of the Thomas-Fermi type*, by Dr. G. H. Wannier, State University of Iowa, introduced by Professor E. N. Oberg.

The speaker discussed differential equations of the type

$$x^2\frac{d^2\phi}{dx^2} - (n + m - 1)x\frac{d\phi}{dx} + nm\phi = rx^q\phi^{1+p}.$$

Such equations occur in the study of potential distributions in the presence of space charge. Particular cases are the equation of Thomas-Fermi,

$$\frac{d^2\phi}{dx^2} = \phi^{3/2}x^{-1/2},$$

and the equation of Langmuir,

$$\frac{d^2\phi}{dx^2} + \frac{1}{x}\frac{d\phi}{dx} = x^{-1}\phi^{-1/2}.$$

11. *Second solutions of certain differential equations associated with the theory of orthogonal polynomials*, by Professor L. W. Swanson, Coe College.

In solving a certain differential equation associated with orthogonal polynomials, a second solution had been omitted. The paper dealt with this second solution of the differential equation.

12. *Mathematics teaching procedure in the light of our experience with the army and navy schools*, by Professor O. C. Kreider, Iowa State College, Professor T. A. Bancroft, Iowa State College, and Professor W. M. Davis, Cornell College.

The speakers discussed the purpose, organization, difficulties, and successes of the army university centers and the navy educational programs overseas.

FRED ROBERTSON, *Secretary*

ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-ninth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Colorado, Boulder, Colorado, on April 19 and 20, 1946.

The attendance was one hundred and twenty, including the following twenty-three members of the Association: H. H. Alden, C. F. Barr, William Betz, J. R. Britton, A. G. Clark, J. R. Everett, J. C. Fitterer, H. T. Guard, D. F. Gunder, Marian S. Gysland, Leota C. Hayward, I. L. Hebel, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, A. J. Lewis, M. L. Madison, A. E. Mallory, W. K. Nelson, Greta Neubauer, O. H. Rechard, A. W. Recht, G. A. Whetstone.

The following papers were presented:

1. *Spherical trigonometry by projection on a plane*, by Professor I. L. Hebel, Colorado School of Mines.

2. *A compatibility relation in the flow of an incompressible ideal fluid*, by Dr. G. A. Whetstone, Amarillo College.

By applying the procedures developed by Riquier for the study of partial differential equations to the usual four equations

$$\frac{\partial u_i}{\partial t} + u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3} = g_{x_i} - \frac{1}{\rho} \frac{\partial P}{\partial x_i}, \quad (i = 1, 2, 3)$$

and

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0,$$

under the assumption ρ = constant, the author was lead to a necessary and sufficient compatibility condition.

3. *Teaching mathematics in the army*, by Professor H. T. Guard, Colorado State College.

This paper consisted of a description of the curriculum and the methods of instruction in the United States Military Academy.

4. *On the definition of functions of a complex variable*, by Professor A. J. Kempner, University of Colorado.

5. *Tables for the power function for tests of hypotheses relating to Poisson distributions*, by Professor A. G. Clark, Colorado A. and M. College.

The speaker discussed devices, including recursion formulas, which serve to reduce the labor of computation in constructing tables for the function specified in the title. Such tables are useful in the construction of efficient sampling experiments.

6. *The present educational situation and the crisis in mathematics*, by William Betz, Public Schools of Rochester, N. Y.

The speaker rehearsed the role of mathematics in the recent war effort. He referred to the mathematical deficiencies of millions of our young men, first pointed out by Admiral Nimitz, and later substantiated by selective service tests. It was suggested that there be a re-examination of the controversy between "education" and mathematics. On the basis of significant quotations it was shown that the educational scene is one of confusion bordering on chaos. Mathematics cannot be adjusted to the prevailing educational philosophies without giving up its real purposes. Fortunately, a healthy reaction against the destructive forces in our educational policies is now in the making. In conclusion, the speaker outlined the remedial steps that seem to be necessary if we wish to help improve the situation.

7. *The Laplace transformation*, by Professor J. R. Britton, University of Colorado.

Professor Britton gave an expository talk on the Laplace transformation and its applications to the solution of boundary value and initial value problems. Some of the simpler transforms were derived, and application was made to the problem of a two mass, two spring vibrating system. A mechanical model

served to demonstrate the types of behavior indicated by the previously obtained solution.

8. *The necessary reconstruction of mathematics in the light of war experiences*, by William Betz.

This was an invited address, delivered at a joint session with the National Council of Teachers of Mathematics and the Mathematics Section of the Colorado Education Association. The speaker holds the position of specialist in mathematics for the public schools of Rochester, N. Y. The address dealt with the reports of various committees which have issued pronouncements on the problem of mathematical instruction in the post-war period. He presented a check list of mathematical objectives, and suggested methods for attaining these objectives.

J. R. BRITTON, *Secretary*

ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The twenty-third annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Louisiana Polytechnic Institute, Ruston, Louisiana, on Friday and Saturday, March 22 and 23, 1946. Professor I. C. Nichols was elected temporary chairman and presided at the Friday afternoon and Saturday morning sessions. Professor P. K. Smith presided at the dinner meeting.

There were fifty in attendance, including the following twenty-one members of the association: W. G. Banks, N. A. Court, J. C. Currie, W. L. Duren, Jr., L. M. Garrison, F. C. Gentry, R. V. Guthrie, Jr., J. A. Hardin, W. L. Johnson, H. T. Karnes, C. G. Killen, A. C. Maddox, Dorothy McCoy, B. E. Mitchell, I. C. Nichols, W. V. Parker, P. K. Rees, F. A. Rickey, H. F. Schroeder, C. D. Smith, H. L. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker, Marelena White.

At the business meeting the following officers were elected for the coming year: Chairman, W. V. Parker, Louisiana State University; Vice-Chairmen, W. L. Johnson, Mississippi Southern College, Z. L. Loflin, Southwestern Louisiana Institute; Secretary-Treasurer, F. C. Gentry, Louisiana Polytechnic Institute. Invitations to meet at Mississippi Southern College in 1947, and at Southwestern Louisiana Institute in 1948 were accepted.

The following papers were presented at the Friday afternoon program:

1. *Esthetic and moral implications of the ark of the covenant*, by Professor B. E. Mitchell, Millsaps College.

The purport of the speaker's remarks was that if the golden rectangle (length/width = 1.618) and the Platonic rectangle (length/width = 1.732) have esthetic value as polygonal forms, then the Mosaic rectangle (length/width = 1.667) has also, since it differs from the arithmetic, geometric, and harmonic means of the other two by 0.008, 0.007, and 0.006 of a part, respectively.

2. *The smoothing effects of linear oscillators*, by Professor W. L. Duren, Tulane University.

It was recalled that forced linear oscillators, which can be realized as simple *R-C-L* electrical circuits, or as mechanical couplings having mass, spring, and dashpot, have a solution whose principal term may be regarded as a smoothed and damped response to the input voltage or force. In contrast to the usual elementary treatment which admits a forcing function which is either a constant or $A \sin \omega t$, the speaker discussed by elementary and intuitive methods the nature of the separate smoothing effects due to inertia, capacitance, and resistance when the forcing function is general in form.

3. *The future importance of secondary mathematics is now in the hands of the college teacher*, by Professor H. T. Karnes, Louisiana State University.

Professor Karnes discussed contemporary problems in the field of secondary mathematics. He commented upon the scarcity of qualified teachers and the shaky place of mathematics in the curriculum. Suggestions were offered for the improvement of the position of mathematics in the schools, and the responsibilities of college teachers in this connection were pointed out.

4. *The presentation of certain application problems in the calculus*, by Professor P. K. Smith, Louisiana Polytechnic Institute.

Professor Smith stated that definitions of mechanical terms in calculus texts are often so presented that the student fails to appreciate the basic physics involved. His paper was primarily concerned with the need of requiring students to understand the physical meaning of the center of mass. A simple procedure based on the law of levers was demonstrated as a means of enabling students to appreciate the physical significance of the center of mass.

5. *Topics in applied mathematics*, by Professor C. D. Smith, Mississippi State College.

In this address it was remarked that editors of magazines should place emphasis both on new mathematics and new uses of mathematics. A recent article, *Mathematics of a Nut Cutter*, in the *National Mathematics Magazine* was used as an illustration.

Professor N. A. Court of the University of Oklahoma was present as the guest of the Section and the Branch of the National Council. At the joint dinner Friday evening he spoke on *The Motionless Arrow*, the address being a discussion of the history and implications of Zeno's paradox. This paper is to be published in the *Scientific Monthly*. At the Saturday morning meeting Professor Court delivered a paper entitled *On a Pencil of Ruled Quadrics*, which was later published in the *Duke Mathematical Journal*.

F. C. GENTRY, *Secretary*

FALL MEETING OF THE INDIANA SECTION

The twenty-fourth annual meeting of the Indiana Section of the Mathematical Association of America was held at Indiana State Teachers College, Terre Haute, Indiana, on Friday, October 18, 1946, in conjunction with the fall meeting of the Indiana Academy of Science. Professor W. L. Ayres presided.

Thirty-four persons registered at the meeting, including the following sixteen members of the Association: W. L. Ayres, Juna L. Beal, W. H. Carnahan, G. E. Carscallen, Olive M. Draper, W. E. Edington, P. D. Edwards, Rufus Isaacs, M. W. Keller, J. P. LaSalle, P. M. Pepper, J. C. Polley, M. E. Shanks, W. O. Shriner, F. C. Smith, and C. P. Sousley.

At the business meeting the following officers were elected for the coming year: Chairman, G. H. Graves, Purdue University; Vice-chairman, H. E. Wolfe, Indiana University; Secretary-Treasurer, M. W. Keller, Purdue University. It was decided to hold a spring meeting in 1947 at a time and place to be determined by the officers.

The following papers were read:

1. *The American University at Shrivenham Barracks*, by Professor P. D. Edwards, Ball State Teachers College.

Shrivenham American University, although created by army personnel awaiting redeployment, was a true American University operated on foreign soil. The faculty of more than 220 members represented 149 American institutions of higher education. About 150 were civilians who were sent to England for this purpose. The University was divided into eight sections which correspond to the usual division of an American university into schools. The faculty of the mathematics branch included fourteen civilians and seven members of the army, all of whom were college teachers in civil life. The enrollment was approximately 4000 each term. Approximately three-fourths of the student body had had actual combat experience. An elective system prevailed, and under it the mathematics branch was exceeded in size by only one other branch. In spite of the unusual difficulties which prevailed, very gratifying results were obtained.

2. *The American University at Biarritz*, by Professor J. C. Polley, Wabash College.

The speaker discussed the mathematics program in the American University at Biarritz, and his experiences while there.

3. *Applications of the linear transformation*, by Professor J. P. LaSalle, Notre Dame University.

Several applications of the linear transformation $(az+b)/(cz+d)$ to problems in electrical engineering were presented by Professor LaSalle. A clear geometric picture of the variation of power transfer to a load with change of load or generator impedance, and of the condition for maximum power transfer, was obtained by means of the transformation $(1-z)/(1+z)$. The "circle" diagram

which relates impedance to the reflection coefficient can be used for this purpose. Though the concepts of reflection and transmission coefficients appear to be more natural than those of impedance and the resulting equivalent circuits, particularly for wave guides, only limited use of the former concepts have been made. This may be due to difficulties in applying the general linear transformation. Algebraic identities which simplify the application of this transformation were given.

4. *The force of mortality function*, by Dr. F. C. Smith, Lincoln Life Insurance.

In this paper, the author discussed the definition of the force of mortality function μ, and some of its properties. Several methods of approximating the values of this function were also presented. The importance of this function in the field of actuarial mathematics was stressed, and the effects of assuming the Gompertz and Makeham hypotheses were shown.

5. *Recent progress in the theory of compressible fluids*, by Professor Rufus Isaacs, Notre Dame University.

Recent developments make the need for a workable theory of compressible fluids imperative. In the past, progress has been checked, first, by the complexity of the theory, and, second, by the formidable amount of numerical computation needed to apply what theory is extant. The new approach of Bergman to the methods of Chaplygin now yields a usable theory when used in conjunction with such modern computational devices as the Aiken machine at Harvard University. A research program under Professor Von Mises is now under way at Harvard.

In two-dimensional incompressible flows, the stream function (a function whose values completely determine the flow) satisfies the Laplace equation. Thus each flow pattern can be determined from an analytic function of a complex variable by taking the imaginary part. In distinction, for compressible fluids the differential equation satisfied by the stream function is non-linear. But Chaplygin showed that in the hodograph plane (where the velocity components are the independent variables) the equation becomes linear although complicated. Bergman has developed an operator for this equation, wherein a flow can again be obtained for each analytic function. This operator requires knowledge of a certain function sequence which may be (and now is being) calculated once and for all. With this apparatus, all flow patterns may be obtained with comparatively little labor.

6. *The achievement of large classes in mathematics* (preliminary report), by Professors H. F. S. Jonah, and M. W. Keller, Purdue University.

The authors discuss in this paper the achievement of large classes in mathematics in comparison with achievement of small classes, as measured by uniform objective tests. These preliminary results indicate that for mature groups and selected instructors, large classes are as effective as small classes for teaching algebra and trigonometry.

7. *Hodograph methods for compressible flow*, by Professor M. E. Shanks, Purdue University.

Professor Shanks discussed the types of hodographs obtainable from flow past an airfoil, and pointed out problems unsolved even for incompressible flows. The case of supersonic flows and the method of characteristics were also discussed.

8. *Engineering applications of spherical trigonometry*, by Professor P. M. Pepper, Notre Dame University.

In this paper Professor Pepper describes some of the engineering applications of spherical trigonometry. Since its inception, spherical trigonometry has been applied principally to the sciences of astronomy, geodesy and navigation. It is little known that spherical trigonometry can be useful to the tool engineer, first, to derive the usual formulas for "compound angles" and, second, to solve atypical problems of this nature. Certain of the compound angle formulas are identified with Napier's rules for right spherical triangles, whereas certain of the nonstandard problems lead to the laws of oblique spherical triangles.

M. W. KELLER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-ninth Summer Meeting, New Haven, Conn., September 1–2, 1947.
Thirty-first Annual Meeting, Athens, Georgia, January 1, 1948.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 24, 1948
OHIO
OKLAHOMA
PACIFIC NORTHWEST
PHILADELPHIA, Bryn Mawr, November 29, 1947
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Redlands, March 13, 1948
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Madison, May, 1947

Outstanding New Editions
By WILLIAM L. HART

COLLEGE ALGEBRA, THIRD EDITION

This revision retains all of the special features which made the previous editions so successful. Contains many fresh exercises; has an attractive new format. *Ready this spring.*

BRIEF COLLEGE ALGEBRA, REVISED

A rapid review of elementary algebra and a thorough treatment of classical college algebra. Has many new exercises. Reset in a fresh, attractive format. *In press.*

INTRODUCTION TO COLLEGE ALGEBRA, REVISED

A review of elementary algebra and a complete, mature treatment of intermediate algebra. Has five new chapters, new exercises, and a new format. *In press.*

Present editions of the Algebras will be available for next year's classes.

MATHEMATICS OF INVESTMENT, THIRD EDITION

A first course in the theory and applications of annuities certain and the mathematical aspects of life insurance. This edition has been particularly designed to adapt the major sections of the material to the needs and the ability of the typical student in a college of business administration. *312 pages (275 pages of text); $3.00. With tables: 440 pages (275 pages of text); $4.00. Tables separately: 128 pages; $1.60.* Also bound with ESSENTIALS OF COLLEGE ALGEBRA: *704 pages (508 pages of text); $4.75.*

D. C. HEATH AND COMPANY

Boston New York Chicago Atlanta San Francisco Dallas London

FORTHCOMING TEXTS

College Algebra

By E. RICHARD HEINEMAN

Associate Professor of Mathematics, Texas Technological College

This new text for freshman students contains an exceptionally thorough review of high school algebra, and introduces very little analytics and almost no calculus. The explanations of factoring and of operations with zero are unusually clear and explicit, as is the explanation of the troublesome question of the "sign of the fraction." The book contains extensive sets of carefully graded problems. *To be published in the summer.* $3.00 (probable)

The Mathematical Theory of Finance

By KENNETH P. WILLIAMS

Professor of Mathematics, Indiana University

This text places emphasis on clear thinking, on thorough analysis of problems to find the simplest method of computation, and on thinking out solutions rather than searching for special formulae. Algebraic problems as well as numerical problems are given, and investment functions are included in the tables of logarithms. *Revised Edition to be published in the summer.* $4.50 (probable)

Elements of Symbolic Logic

By HANS REICHENBACH

Professor of Philosophy, University of California at Los Angeles

Beginning with a general exposition of the problems of logic and language, this text then takes up the calculus of propositions, for which the truth table method is used. It then turns to the calculus of functions, for which the truth table method is developed. Further chapters deal with the calculus of classes, of the higher calculus of functions, conversational language, and connective operations and modalities. *To be published in the summer.* $4.50 (probable)

THE MACMILLAN COMPANY

TEXTBOOK NEWS

SOLID ANALYTIC GEOMETRY

By

John M. H. Olmstead, Ph. D.

University of Minnesota

THIS new textbook, which presents a complete and lucid treatment of analytic geometry of three dimensions, is designed for courses on the junior-senior or early graduate level. The book contains the material for a very rich and extensive course, so arranged as to be conveniently adapted to courses of various content. Other features of the book are the simplicity of the presentation, the emphasis on logical reasoning and method, the many illustrative examples, over one thousand exercises, the systematic elementary treatment of Matrix Algebra, and the clarity of the fifty line drawings, representing surfaces and other spatial objects. *To be published in May.*

D. APPLETON-CENTURY COMPANY
35 West 32nd Street New York 1, New York

PRINCETON MATHEMATICAL SERIES

Edited by MARSTON MORSE H. P. ROBERTSON A. W. TUCKER

1. The Classical Groups. By Hermann Weyl	314 pages, $5.00
2. Topological Groups. By L. Pontrjagin (translated by E. Lehmer)	310 pages, $5.00
3. An Introduction to Differential Geometry. By Luther Pfahler Eisenhart	316 pages, $3.50
4. Dimension Theory. By Witold Hurewicz & Henry Wallman	174 pages, $3.00
5. Analytical Foundations of Celestial Mechanics. By Aurel Wintner	460 pages, $6.00
6. Laplace Transform. By David Vernon Widder	416 pages, $6.00
7. Integration. By Edward James McShane	400 pages, $6.00
8. Theory of Lie Groups. By Claude Chevalley	228 pages, $3.00
9. Mathematical Methods of Statistics. By Harold Cramér	570 pages, $6.00
10. Several Complex Variables. By S. Bochner & W. T. Martin	(In preparation)
11. Introduction to Topology. By S. Lefschetz	(In preparation)

The prices above are subject to an educational discount of fifteen per cent

PRINCETON UNIVERSITY PRESS, PRINCETON, N.J.

Two New Algebra Texts

by JACK R. BRITTON

Associate Professor, Engineering Mathematics

and

L. CLIFTON SNIVELY

Assistant Professor, Engineering Mathematics
both of the University of Colorado

Algebra for College Students

Starting with the fundamental ideas of arithmetic and elementary algebra, this comprehensive text develops the subject of algebra completely. The explanations are simple and deliberate and full enough to meet the needs of every student.

The first half of the book supplies an adequate course in intermediate algebra. The last part develops the topics of the customary course in college algebra. Throughout the book, the major emphasis is on the *important underlying ideas.*

Among the features of the new text are: more geometrical material than is usual in an algebra book; use of the practical significance of variation as met in the laboratory; a chapter devoted to certain aspects of computation with approximate numbers; emphasis on the fact that corresponding logarithms to different bases are proportional to each other; a brief exposition of vectors; reversal of the traditional positions of Theory of Equations and Systems Involving Quadratic Equations and emphasis on the general ideas involved in solving any system of equations.

544 pp. $3.00

Intermediate Algebra

Comprised of Chapters 1-12 of *Algebra for College Students,* covering Fundamental Ideas from Arithmetic through Quadratic Equations and Ratio, Proportion, and Variation; with three additional chapters on Logarithms, Progressions and the Binomial Theorem, and Systems Involving Quadratic Equations.

Prob. pp. 320 Prob. date May Prob. price $2.00

Rinehart & Company, Inc.

232 MADISON AVENUE · NEW YORK 16, N. Y.

Just Published!

A New Text by the Author of
Intermediate Algebra for College Students

COLLEGE ALGEBRA

By THURMAN S. PETERSON

Associate Professor of Mathematics, University of Oregon

The many teachers who are finding Peterson's *Intermediate Algebra for College Students* one of the most successful textbooks ever published in the elementary field will welcome the announcement that his new *COLLEGE ALGEBRA* is now ready.

Characterized by the same careful workmanship and effective presentation that have made the earlier text so popular, the new text is designed for students who have had at least one year of high school algebra. It is full enough and rigorous enough to provide a foundation course which prepares for advanced college mathematics, but at the same time it is so planned that it can be adapted equally well to the terminal algebra course for non-scientific students.

COLLEGE ALGEBRA is characterized by clear, thorough, precise presentation; systematic, concise organization of topics; numerous illustrations and examples; many informational notes; a great abundance of carefully graded problems—a total of 3140, of which nearly 400 are *stated* problems.

Special features include a number of new computational methods; emphasis on applications of the subject wherever possible; a final chapter on statistical methods which contains material new to algebra texts.

Answers to odd-numbered problems are included in the text. Answers to even-numbered problems are contained in a separate pamphlet which is available without charge at the request of the instructor.

334 pages *Price $2.50*

HARPER & BROTHERS, 49 East 33d Street, New York 16

ADVANCED CALCULUS

By *David V. Widder, Harvard University*

This significant addition to Prentice-Hall Mathematics texts offers two welcome innovations in subject matter. They are chapters on the Stieltjes Integral and the Laplace Transform. These subjects, never before accessible in elementary form, are becoming increasingly important to mathematicians. An added feature of this new study is the unusually clear treatment of Line Integrals and Green's Theorem.

CALCULUS

By *Lyman M. Kells, United States Naval Academy*

Two conspicuous features of this text illustrate its distinctiveness:

- Makes the fullest use of review as an aid to learning, first acquainting the student with a technique simply by inspection, later returning to it for a more formal discussion.
- The author actually talks with the student. Even simple things are explained down to the smallest detail.

Exceptionally thorough explanations and a freshness and abundance of problems provide a remarkably improved approach to the calculus.

ANALYTIC GEOMETRY

By *Francis D. Murnaghan, Johns Hopkins University*

"One of the most refreshing and brilliantly organized books on a mathematical subject to appear in years."—*Lloyd Lassen, North Texas Agricultural College*

Send for your approval copies

MATHEMATICS OF FINANCE

SECOND EDITION, REVISED AND ENLARGED

By *Thomas M. Simpson, Zareh M. Pirenian and Bolling H. Crenshaw*

Here is the best possible preparation for students intending to enter business. Now brought completely up-to-date, it includes all new problems, enlarged tables, a new chapter on statistics, simplified treatment of general annuities, and enlarged treatment of life insurance and problems of amortization of all types of debts. The text is arranged in two parts, which may be obtained either in a single volume, or in two volumes for single-semester courses.

COLLEGE ALGEBRA

WITH REVISIONS AND ADDITIONAL PROBLEMS

By *Albert Thayer Davis, Northwestern University*

"The book is clear, interesting, stimulating. Especially welcome are the historical and philosophical comments."—*J. L. Walsh, Harvard University*

In 16 chapters it covers the standard algebra course. Four additional chapters contribute important material on statistics, the history and uses of mathematics, and "mathematical reactions"—unusual topics which add tremendously to the interest and usefulness of a college text. Provides approximately 3,000 problems.

MATHEMATICS IN HUMAN AFFAIRS

By *F. W. Kokomoor, University of Florida*

An unusual and absorbing text combining the history and study of mathematics. The subject is presented as an exciting narrative, and explanations are simple enough for the student with almost no previous training. Fully illustrated.

Send for your approval copies

PRENTICE-HALL, INC. 70 FIFTH AVENUE, NEW YORK 11

Distinctive McGraw-Hill Books

Differential and Integral Calculus

By Ross R. MIDDLEMISS, Washington University. Second edition. 497 pages, $3.25

Planned to give the student a real understanding of the principal concepts of the subject, this book maintains a high degree of accuracy, and is at the same time exceptionally clear and readable.

Analytic Geometry

By Ross R. MIDDLEMISS. 306 pages, $2.75

Designed especially to meet the needs of scientific and technical students in preparing them for calculus and the various sciences.

Plane and Spherical Trigonometry

By LYMAN M. KELLS, WILLIS F. KERN and JAMES R. BLAND, United States Naval Academy. Second edition. 401 pages, $2.25. With tables, 516 pages, $3.00. Tables alone, $1.00

A simple, logically arranged text in which the topics are so spaced that each is mastered in its turn.

Elementary Differential Equations

By LYMAN M. KELLS. Third edition. In press

Features: simple presentation of the fundamental types of differential equations; numerous illustrative examples; and abundant exercises and applications. All material has been clarified, expanded, and strengthened.

Six-place Tables

By EDWARD S. ALLEN, Iowa State College. Seventh edition. 195 pages, $2.50

A selection of tables of squares, cubes, square roots, fifth roots and powers, circumferences and areas of circles, common logarithms of numbers, etc.

Essential Business Mathematics

By LLEWELLYN R. SNYDER, San Francisco Junior College. 461 pages, $2.75

A collegiate text in arithmetic, presenting the fundamentals of business mathematics, including refresher work in computation, etc.

Algebra. A Second Course

By R. ORIN CORNETT, Oklahoma Baptist University. 313 pages, $2.20

Based on the idea that mastery of detail and routine operations should be preceded by an understanding of the purpose, significance, and relation to the whole.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc.

330 West 42nd Street **New York 18, N.Y.**

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

6

NOLOGY DEPARTMENT

PUBLIC LIBRARY
JUL 1 1947
DETROIT 8

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 54 NUMBER 6

CONTENTS

Benjamin Franklin Finkel . . . . . W. D. CAIRNS 311
The New Secretary-Treasurer 312
Remarks on Analyticity and Integration . . . . . A. D. PERRY AND J. W. T. YOUNGS 313
Colin Maclaurin . . . . . H. W. TURNBULL 318
The Teaching of College Mathematics Today . M. S. KNEBELMAN 323
The Hervey Point of the General n-Line . . . R. GOORMAGHTIGH 327
Mathematical Notes . . . . . V. L. KLEE, WILLIAM GUSTIN 332
Classroom Notes . . . . . C. C. MACDUFFEE, M. K. FORT 335
Elementary Problems and Solutions 339
Advanced Problems and Solutions 345
Recent Publications 353
Clubs and Allied Activities 355
News and Notices 358
The Mathematical Association of America 361
Annual Meeting of the Illinois Section 361
November Meeting of the Philadelphia Section 363
Fall Meeting of the Maryland-District of Columbia-Virginia Section 364
New Members 366
Calendar of Future Meetings 368

JUNE–JULY 1947